The Restriction and Kakeya Conjectures

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Abstract

We survey the progress made on the restriction problem since it was first conjectured in the 1960s by E. M. Stein, in particular the oscillatory-integral approach which culminated in the Tomas-Stein theorem of 1975. We also examine the connections between the restriction and Kakeya problems, the latter evolving from a problem posed by S. Kakeya in 1917. In particular we devise a correspondence between the restriction and Kakeya set conjectures which is able to compare progress on the two problems in a quantitative way. Finally we discuss the latest developments which rely on bilinear, and their natural extension, multilinear, estimates and which have been found to provide the best known results for their linear counterparts (i.e. on the original problems).

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Chapter 1

Introduction

The restriction conjecture originated in the 1960s with an observation of Elias Stein [22]. He noted that for certain zero-measure sets S, which possess sufficent curvature, the Fourier transform of L^p functions can be 'restricted' for certain $1 \leqslant p < 2$.

By 'restricted' we mean restricting the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\mathrm{d}x$$

to a subset S of $\mathbb{R}^n$. The resulting function we will denote by $\widehat{f}|_S$.

By the Riemann-Lebesgue lemma, if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is a continuous, bounded function on $\mathbb{R}^n$ which vanishes at infinity. This means that we can meaningfully restrict this function to any $S \subseteq \mathbb{R}^n$ (in particular, to the sets that we are interested in, namely those of measure zero) in that $\widehat{f}|_S$ will have finite L^q norm for any $1 \leqslant q \leqslant \infty$. In other words, we have $\|\widehat{f}\|_{L^q(S)} \leqslant C\|f\|_{L^1(\mathbb{R}^n)}$ for all $1 \leqslant q \leqslant \infty$, where C is a constant.

However, if $f \in L^2(\mathbb{R}^n)$ then, by Plancherel's theorem $\widehat{f}(\xi) \in L^2(\mathbb{R}^n)$ also, so there is no meaningful way to restrict $\widehat{f}(\xi)$ to a set of measure zero. This means that we do not have $\|\widehat{f}\|_{L^q(S)} \leqslant C\|f\|_{L^2(\mathbb{R}^n)}$ for any q. In fact this inequality does not even make sense, let alone hold, because the left side is not defined.

Of course, we could restrict the Fourier transform to a set of non-zero measure, for instance the unit ball $B(0,1)$. From the Hausdorff-Young inequality we have

$$\|\widehat{f}\|_{L^q(B(0,1))} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$$

for all $q \leqslant p'$ and $1 \leqslant p \leqslant 2$. What Stein noticed, however, was that the situation is more interesting when $\widehat{f}$ is restricted to a set of zero measure, but not for any set of zero measure: given a hyperplane a function can be found which lies in L^p for every $p > 1$ but which has infinite Fourier transform on every point of the hyperplane. For instance, the function

$$f(x) = \frac{\psi(x_2, \ldots, x_n)}{1 + |x_1|}$$

where ψ is a bump function, is such a function whose Fourier transform is infinite on the hyperplane $\{\xi \in \mathbb{R}^n : \xi_1 = 0\}$. So we can not meaningfully restrict the Fourier transform to a hyperplane or even to a compact subset of a hyperplane.

It is when a zero measure set has some curvature that we are able to find some non-trivial 'restriction estimates' of the form

$$\|\widehat{f}\|_{L^q(S)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

There are, of course, infinitely many such sets but the ones which have been investigated the most are the hypersurfaces

$$S_{\text{sphere}} = \{\xi \in \mathbb{R}^n : |\xi| = 1\},$$

$$S_{\text{parabola}} = \{\xi \in \mathbb{R}^n : \xi_n = \frac{1}{2}|\xi|^2\},$$

and

$$S_{\text{cone}} = \{\xi \in \mathbb{R}^n : \xi_n = |\xi|\}.$$

In this investigation we will mostly be concerned with the first of these, henceforth denoted by S^{n-1}, primarily because it is the simplest compact, co-dimension 1 submanifold with non-vanishing Gaussian curvature. It is believed that the conjectured range of exponents for which the Fourier transform can be meaningfully restricted for S^{n-1} is the same as for any compact hypersurface whose Gaussian curvature is always non-vanishing.

The question which then naturally arises is: what happens for $1 < p < 2$? This is the restriction problem, and the conjectured answer is that we have

$$\|\widehat{f}\|_{L^q(S^{n-1})} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$$

for

$$p' \geqslant \frac{n+1}{n-1}q \qquad \text{and} \qquad p < \frac{2n}{n+1}.$$

This has been shown for $n = 2$ [32] and [16] but is still an open problem for $n \geqslant 3$. In this thesis we will be investigating the origin of the conjectured bounds and some of the partial progress achieved in proving them. To describe all of the work that has been done on this problem would be too ambitious an aim for this MPhil thesis so we restrict our attention to the oscillatory-integral methods culminating in the Tomas-Stein theorem of the 1970s [28]. This establishes the bounds

$$q = 2, \qquad 1 \leqslant p \leqslant 2\frac{n+1}{n+3}.$$

More recent work has involved wave-packet decomposition and has achieved [26]

$$p' \geqslant \frac{n+1}{n-1}q, \qquad p' > \frac{2(n+2)}{n}.$$

We will also investigate the geometric-combinatorial methods employed in the more recent work of Wolff, Bourgain, Tao (see, for instance [27], [11] and [29]) and others which has been motivated by the, at first sight surprising, connection between the restriction problem and this problem posed in 1917 by Kakeya [5]:

> *"In the class of figures in which a segment of length 1 can be turned around 360°, remaining always within the figure, which one has the smallest area?"*

In 1920 Besicovitch solved this problem [5] by showing that one can have such a figure, called a Besicovitch set, with a line segment in every direction, with arbitrarily small measure. Further, he showed this for all dimensions $n \geqslant 2$. However, the Kakeya set conjecture:

> *"Let $E \subseteq \mathbb{R}^n$ be a Besicovitch set. Then $\dim(E) = n$."*

(where $\dim(E)$ is the Minkowski dimension, d, defined by $\lim\limits_{\delta \to 0} \log_\delta |E_\delta| = n - d$, where E_δ is the δ-neighbourhood of E), remains an open problem for $n \geqslant 3$.

There is an intermediate conjecture, called the Kakeya maximal operator conjecture, which is implied by the restriction conjecture and which, in turn, implies the Kakeya set conjecture [24], and, as such, both Kakeya conjectures have been shown for $n = 2$. The Kakeya maximal operator conjecture is concerned with the control of the overlap of a family of tubes of equal size but whose directions form a δ-net of the unit sphere S^{n-1} (where δ is a small parameter). Since the tubes belong to $\mathbb{R}^n$ but their directions to S^{n-1} we have an indication that this problem might be connected to the restriction problem.

There has been significant progress in the last 20 years on the Kakeya maximal operator conjecture, and hence the set conjecture. The most significant breakthroughs have been due to Bourgain's 'bush' [6] and Wolff's 'hairbrush' [30] arguments, the latter being a refinement of the former. The 'bush' argument relies on the fact that a collection of tubes of differing orientations but containing a common point have diminishing overlap away from that point, and results in $\dim(E) \geqslant \frac{n+1}{2}$. The 'hairbrush', meanwhile, utilises the disjointness properties of such a collection of tubes which pass through a common *line*, and results in $\dim(E) \geqslant \frac{n+2}{2}$.

We will reproduce both of these arguments (the 'hairbrush' being postponed until we have introduced bilinear estimates) and then examine the correspondence between the respective partial progress made on the restriction and Kakeya set conjectures. Since the former implies the latter this will provide us with a way to compare what information is provided about the set conjecture from the 'direct' geometrical methods with that implied by known restriction estimates.

In chapter 4 we discuss the bilinear approach to the restriction and Kakeya conjectures. Here the concept of *transversality* (as defined in section 4.1), as well as curvature, is of central importance (see [25] or [3]). This manifests in the restriction problem by considering restricting the Fourier transform of functions on two caps contained in S^{n-1}, whose normal vectors are sufficiently separated in direction, simultaneously, and in the Kakeya problem by considering *two* families of tubes, the members of each of which have orientations which are sufficiently close to a pair of fixed, linearly independent basis vectors of S^{n-1}. There exist bilinear, and, indeed, arbitrarily high-dimensional multilinear, analogues of the conjectures discussed above and these are responsible for the most recent progress on the linear versions of the conjectures. In chapter 5 we finish by discussing some of the latest developments in the field.

Chapter 2

The Restriction Conjecture

In this chapter we will be looking at the justification of the bounds in the restriction conjecture. In other words, the best exponents p and q that there can possibly be. We will then go on to discuss the culmination of the oscillatory-integral approach to the problem in the 1970s, which was the Tomas-Stein theorem. The ideas expressed in this chapter are based on those found in [24].

2.1 The Origin of the Bounds in the Restriction Conjecture

The restriction conjecture states that $\|\widehat{f}\|_{L^q(S^{n-1})} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$ only when $p' \geqslant \dfrac{n+1}{n-1}q$ and $p < \dfrac{2n}{n+1}$, where S^{n-1} is the unit sphere.

2.1.1 The $p' \geqslant \frac{n+1}{n-1}q$ bound

Let S be a surface: $S = \{(\underline{x}, \Phi(\underline{x})) : \underline{x} \in \mathbb{R}^{n-1}, |\underline{x}| \lesssim 1\}$ where $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function such that $\Phi(0) = \nabla\Phi(0) = 0$.

Proposition 2.1 *Suppose Φ vanishes to order k at 0 for some $k \geqslant 2$, so that $\Phi(x) = O(|x|^k)$. Then $\|\widehat{f}\|_{L^q(S)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$ only when $p' \geqslant \dfrac{n+k-1}{n-1}q$. In particular $p' \geqslant \dfrac{n+1}{n-1}q$ is necessary for any S.*

Proof. Let ψ be a Schwartz function such that $\widehat{\psi} \sim 1$ near the origin. Let $f(x_1, \ldots, x_{n-1}, x_n) = \psi\left(\frac{x_1}{\lambda^{\frac{1}{k}}}, \ldots, \frac{x_{n-1}}{\lambda^{\frac{1}{k}}}, \frac{x_n}{\lambda}\right)$ for some $\lambda \gg 1$. Observe that $\|f\|_{L^p} \approx \lambda^{\frac{(n+k-1)}{kp}}$. (For instance if ψ is a Gaussian then

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(e^{-\left(\frac{x_1}{\lambda^{\frac{1}{k}}}\right)^2 - \left(\frac{x_2}{\lambda^{\frac{1}{k}}}\right)^2 - \ldots - \left(\frac{x_n}{\lambda}\right)^2}\right)^p\right)^{\frac{1}{p}}$$

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} e^{-p\left(\frac{x_1}{\lambda^{\frac{1}{k}}}\right)^2 - p\left(\frac{x_2}{\lambda^{\frac{1}{k}}}\right)^2 - \ldots - p\left(\frac{x_n}{\lambda}\right)^2}\right)^{\frac{1}{p}}$$

$$\sim (\underbrace{\lambda^{\frac{1}{k}}.\lambda^{\frac{1}{k}}\ldots\lambda^{\frac{1}{k}}}_{\text{n-1 terms}}.\lambda)^{\frac{1}{p}} = \lambda^{\frac{n-1+k}{kp}}.)$$

The Fourier transform of this is

$$\widehat{f}(\xi_1, \ldots, \xi_{n-1}, \xi_n) = \lambda^{\frac{(n+k-1)}{k}} \widehat{\psi}(\lambda^{\frac{1}{k}}\xi_1, \ldots, \lambda^{\frac{1}{k}}\xi_{n-1}, \lambda\xi_n).$$

By the hypothesised conditions on S we see that S contains a 'cap' of radius $\sim \lambda^{-\frac{1}{k}}$ and surface measure $\lambda^{-\frac{n-1}{k}}$. If we restrict $\widehat{f}$ to S we see that $\widehat{f} \sim \lambda^{\frac{n+k-1}{k}}$ on this cap. So we have

$$\|\widehat{f}\|_{L^q(S)} \geqslant \lambda^{\frac{n+k-1}{k}} \lambda^{-\frac{n-1}{kq}}.$$

So $\|\widehat{f}\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ is only possible if

$$\lambda^{\frac{n+k-1}{k}} \lambda^{-\frac{n-1}{kq}} \lesssim \lambda^{\frac{(n+k-1)}{kp}}.$$

(Where we have used the notation $A \lesssim B$ to represent $A \leqslant CB$ where C is an unspecified constant.) Letting $\lambda \to \infty$ we obtain

$$n+k-1-\frac{n-1}{q} \leqslant \frac{n+k-1}{p},$$

$$(n+k-1)(1-\frac{1}{p}) \leqslant \frac{n-1}{q},$$

$$p' \geqslant q\frac{n+k-1}{n-1}.$$

Since $k \geqslant 2$ we have

$$p' \geqslant \frac{n+1}{n-1}q.$$

□

2.1.2 The $p < \frac{2n}{n+1}$ bound

Extension Theorems

To see where the $p < \frac{2n}{n+1}$ bound comes from we first need to introduce the idea of an extension theorem: suppose that there exists values of p and q such that

$$\|\widehat{f}\|_{L^q(S^{n-1})} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$$

holds. In particular we have

$$\sup_{\|f\|_{L^p}=1} \|\widehat{f}\|_{L^q(S^{n-1})} \lesssim 1. \tag{2.2}$$

If we define $l_{\widehat{f}}$ as the bounded linear functional on $L^{q'}$ given by

$$l_{\widehat{f}}g = |\int g\widehat{f}|$$

we have that $l_{\widehat{f}}$ is the dual space of $\widehat{f}$ and that $\|l_{\widehat{f}}\|_{\text{op}} = \|\widehat{f}\|_{L^q}$. Now

$$\|l_{\widehat{f}}\|_{\text{op}} = \sup_{g\in L^{q'}} \frac{|l_{\widehat{f}}g|}{\|g\|_{L^{q'}}} = \sup_{\|g\|_{L^{q'}}=1} |l_{\widehat{f}}g| = \sup_{\|g\|_{L^{q'}}=1} |\int g\widehat{f}|,$$

so

$$\|\widehat{f}\|_{L^q} = \sup_{\|g\|_{L^{q'}}=1} |\int g\widehat{f}|.$$

and (2.2) becomes

$$\sup_{\|f\|_{L^p}=1} \sup_{\|g\|_{L^{q'}(S^{n-1})}=1} |\int \widehat{f}(\xi)g(\xi)\mathrm{d}\omega(\xi)| \lesssim 1,$$

where $\mathrm{d}\omega$ is the surface measure of the unit sphere.

We can now reverse the order in which we take the two supremums and apply Parseval's theorem:

$$\sup_{\|g\|_{L^{q'}(S^{n-1})}} \sup_{\|f\|_{L^p}=1} |\int f(x)\widehat{g\mathrm{d}\omega}(x)\mathrm{d}x| \lesssim 1.$$

We can also reverse the above step where we utilised duality:

$$\sup_{\|g\|_{L^{q'}(S^{n-1})}} \|\widehat{g\mathrm{d}\omega}\|_{L^{p'}} \lesssim 1,$$

and therefore

$$\|\widehat{g\mathrm{d}\omega}\|_{L^{p'}} \lesssim \|g\|_{L^{q'}}.$$

This is known as an extension theorem. At this point it is convenient to introduce some notation. Let us denote by $R_S(p \to q)$ the restriction estimate

$$\|\widehat{f}\|_{L^q(S;\mathrm{d}\omega)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

and similiarly, by $R_S^*(q' \to p')$ the extension estimate

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(S;\mathrm{d}\omega)}.$$

If we choose $g \equiv 1$, we have $\|\widehat{\mathrm{d}\omega}\|_{L^{p'}} \lesssim 1$. We then use a decay estimate for $\widehat{\mathrm{d}\omega}$, derived from the method of stationary phase (which we will prove in the next subsection):

Proposition 2.3 *If* $\mathrm{d}\omega$ *is the surface measure of the unit sphere, then for* $|x| \gg 1$ *we have*

$$\widehat{\mathrm{d}\omega}(x) = C\frac{e^{2\pi i|x|}}{|x|^{\frac{n-1}{2}}} + C\frac{e^{-2\pi i|x|}}{|x|^{\frac{n-1}{2}}} + O(|x|^{-\frac{n}{2}}).$$

So for $\widehat{\mathrm{d}\omega}$ to be in $L_{p'}$ we need $|x|^{-p'\frac{n-1}{2}}$ to decay faster than $|x|^{-n}$ i.e.

$$p'\frac{n-1}{2} > n,$$

$$p' > \frac{2n}{n-1}.$$

2.1.3 The Method of Stationary Phase

To prove Proposition 2.3 we will first need two lemmas, both dealing with the behaviour of integrals of the form

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\mathrm{d}x$$

(namely oscillatory integrals of the first kind) where λ takes large, positive values, ϕ is a real-valued, smooth function (the phase) and ψ is complex-valued, and smooth. The proofs we provide are based on those found in [23].

First we need the following proposition concerning integrals in one dimension, for $a \leqslant x \leqslant b$

Proposition 2.4 *Let* ϕ *and* ψ *be smooth functions so that* ψ *has compact support in* (a, b)*, and* $\phi' \neq 0$ *for all* $x \in [a, b]$*. then*

$$I(\lambda) = O(\lambda^{-N}) \quad \textit{as } \lambda \to \infty$$

for all $N \geqslant 0$*.*

Proof. Let D denote the differential operator

$$Df(x) = (i\lambda\phi'(x))^{-1} \cdot \frac{\mathrm{d}f}{\mathrm{d}x}$$

and let D^* denote the adjoint operator to D so that $\langle Df, g\rangle = \langle f, D^* g\rangle$.

Now

$$\langle Df, g\rangle = \int_a^b \frac{1}{i\lambda\phi'(x)} f'(x)g(x)\mathrm{d}x,$$

and so, by integration by parts, we have

$$\langle Df, g\rangle = -\int_a^b f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\frac{1}{i\lambda\phi'(x)}\right)\mathrm{d}x,$$

so

$$D^* g(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\frac{1}{i\lambda\phi'(x)}\right)\mathrm{d}x.$$

Since $D^N(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)}$ for every integer N, and $\left\langle D^N f, g\right\rangle = \left\langle f, (D^*)^N g\right\rangle$, if we put $f(x) = e^{i\lambda\phi(x)}$ and $g(x) = \psi(x)$ we have

$$I(\lambda) = \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)\mathrm{d}x = \int_a^b e^{i\lambda\phi(x)}(D^*)^N(\psi(x))\mathrm{d}x,$$

and

$$|I(\lambda)| \leqslant \int_a^b |e^{i\lambda\phi(x)}(D^*)^N(\psi(x))|\mathrm{d}x = \int_a^b |(D^*)^N(\psi(x))|\mathrm{d}x = A_N\lambda^{-N},$$

for some constant A_N, for every N, and the lemma is proved. □

Lemma 2.5 *Principle of non-stationary phase*

Suppose ϕ and ψ are defined as above but with ψ having compact support in $\mathbb{R}^n$ and ϕ having no critical points in the support of ψ where x_o is said to be a critical point if

$(\nabla\phi)(x_o) = 0$. *Then*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x) = O(\lambda^{-N})$$

as $\lambda \to \infty$, *for every* $N \geqslant 0$.

Proof. By hypothesis, at every point x_o in the support of ψ there exists a unit vector η and a small ball $B(x_o)$ centered at x_o such that

$$\eta \cdot (\nabla\phi)(x) \geqslant c > 0$$

for all $x \in B(x_o)$.We can rewrite $I(\lambda)$ as the finite sum

$$\sum_j \int e^{i\lambda\phi(x)}\psi_j(x)\mathrm{d}x,$$

where the ψ_j are smooth and have compact support in one of these $B(x_o)$. We have then reduced the problem to proving the result for each of these integrals. If we choose a co-ordinate system $x_1, \ldots x_n$ such that x_1 lies along η we have

$$\int e^{i\lambda\phi(x)}\psi_j(x)\mathrm{d}x = \int \left(\int e^{i\lambda\phi(x_1,\ldots x_n)}\psi_j(x_1,\ldots x_n)\mathrm{d}x_1 \right) \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

Then by proposition 2.4 we see that the integral is $O(\lambda^{-N})$ and so the result follows. □

Definition 2.6 *A critical point* x_0 *of* ϕ *is said to be* non-degenerate *if the symmetric* $n \times n$ *matrix*

$$\left.\frac{\partial^2\phi}{\partial x_i \partial x_j}\right|_{x_0}$$

is invertible.

Lemma 2.7 *Principle of stationary phase for non-degenerate stationary points.*

Let x_o be a point in $\mathbb{R}^n$. Suppose ϕ is a smooth real function on a neighbourhood of x_o which has a non-degenerate stationary point at x_o, but

$$\det(\partial_i\partial_j\phi(x_o)) \neq 0,$$

where $\partial_i\partial_j\phi(x_o)$ is the Hessian matrix of ϕ at x_o. Then, if ψ is a bump function supported on a sufficiently small neighbourhood of x_o, we have

$$\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)\mathrm{d}x = C\psi(x_o)e^{i\lambda\phi(x_o)}\lambda^{\frac{-n}{2}} + O(\lambda^{\frac{-(n+1)}{2}})$$

as $\lambda \to +\infty$, where C is a constant depending on ϕ.

For the proof of this lemma see [23].

We are now in a position to prove Propn.2.3:

Proof. Propn.2.3

Utilising the radial symmetry of the unit sphere we can put $x = \lambda e_n$ for some $\lambda \gg 1$, where e_n is a unit vector in arbitrary direction. The Fourier transform of the surface measure

$$\widehat{\mathrm{d}\omega}(x) = \int_{S^{n-1}} e^{-2\pi i x\cdot\omega}\mathrm{d}\omega$$

then becomes

$$\widehat{\mathrm{d}\omega}(\lambda e_n) = \int_{S^{n-1}} e^{-2\pi i\lambda\omega_n}\mathrm{d}\omega. \tag{2.8}$$

ω_n is just the size of the projection of the vector ω onto e_n and so takes values in the range [-1,1], is stationary when $\omega = \pm e_n$ and non-stationary otherwise. We thus rewrite the right of (2.8) as

$$\int_{S^{n-1}} e^{-2\pi i\lambda\omega_n}\psi_+(\omega)\mathrm{d}\omega + \int_{S^{n-1}} e^{-2\pi i\lambda\omega_n}\psi_-(\omega)\mathrm{d}\omega + \int_{S^{n-1}} e^{-2\pi i\lambda\omega_n}(1-\psi_+-\psi_-)(\omega)\mathrm{d}\omega, \tag{2.9}$$

where ψ_+ and ψ_- are cutoff functions supported on a smalll neighbourhood of e_n and $-e_n$, respectively.

Let us rewrite ω_n, in terms of $\underline{\omega} = (\omega_1, \ldots \omega_{n-1})$, as

$$\omega_n = (1 - |\underline{\omega}|^2)^{1/2},$$

and call this function $\Phi(\underline{\omega})$. Clearly, $\Phi(\underline{\omega})$ has a non-degenerate stationary point at $\underline{\omega} = \underline{0}$. Then by Lemma 2.7 the contribution of the first term of (2.9) is

$$Ce^{-2\pi i\lambda\Phi(\underline{0})}\lambda^{\frac{-(n-1)}{2}} + O(\lambda^{\frac{-n}{2}}),$$

and since $\Phi(\underline{0}) = 1$ this is equal to

$$Ce^{-2\pi i\lambda}\lambda^{\frac{-(n-1)}{2}} + O(\lambda^{\frac{-n}{2}}).$$

Similarly, the second term of (2.9) contributes

$$Ce^{2\pi i\lambda}\lambda^{\frac{-(n-1)}{2}} + O(\lambda^{\frac{-n}{2}}).$$

By Lemma 2.5 the contribution of the third term in (2.9) is $O(\lambda^{-N})$ for any N. Combining these contributions we arrive at Propn.2.3 $\square$

2.2 The Tomas-Stein Restriction Theorem

Theorem 2.10 *If*

$$1 \leqslant p \leqslant 2\frac{n+1}{n+3}$$

then

$$\|\widehat{f}\|_{L^2(S^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.11}$$

This is known as the Tomas-Stein restriction theorem and is a significant positive result in the history of progress on the full Restriction Conjecture.

2.2.1 The TT^* Method

An operator T is bounded from L^p to L^2 if and only if its square TT^* is bounded from $L^{p'}$ to L^p:

square (2.11),

$$\int |\widehat{f}(\xi)|^2 \mathrm{d}\omega(\xi) \lesssim \|f\|_{L^p}^2,$$

$$\left\langle \widehat{f}, \widehat{f}\mathrm{d}\omega \right\rangle \lesssim \|f\|_{L^p}^2,$$

$$\left\langle \widehat{f}, \widehat{f * \widehat{\mathrm{d}\omega}} \right\rangle \lesssim \|f\|_{L^p}^2.$$

Since the Fourier transform is a unitary operator

$$\left\langle \widehat{f}, \widehat{f * \widehat{\mathrm{d}\omega}} \right\rangle = \left\langle f, \left(\widehat{f * \widehat{\mathrm{d}\omega}}\right)^{\vee} \right\rangle = \left\langle f, f * \widehat{\mathrm{d}\omega} \right\rangle \lesssim \|f\|_{L^p}^2.$$

From Hölder's inequality

$$\|f \cdot f * \widehat{\mathrm{d}\omega}\|_{L^1} \leqslant \|f\|_{L^p} \|f * \widehat{\mathrm{d}\omega}\|_{L^{p'}}$$

so it suffices to prove

$$\|f * \widehat{\mathrm{d}\omega}\|_{L^{p'}} \lesssim \|f\|_{L^p}.$$

2.2.2 Proof of the Tomas-Stein Restriction Theorem using Complex Interpolation

We can very nearly prove the Tomas-Stein restriction theorem using real interpolation, we will only be missing the endpoint $p = \dfrac{2(n+1)}{n+3}$. Due to the limitations on what can be included in a thesis of this scope we refer the reader to [24] for the proof with complex interpolation which achieves the endpoint. This subsection also follows [24].

The fundamental idea that Tomas had in [28] was to break-up $\widehat{\mathrm{d}\omega}$ dyadically: if we define $\phi(x)$ to be a radially symmetric bump function equal to 1 at $x = 0$ with compact support and then

$$\psi_k(x) = \phi(2^{-k}x) - \phi(2^{1-k}x)$$

so that each $\psi_k(x)$ has size 1 and is supported on the annulus $|x| \approx 2^k$, we have

$$1 = \phi(x) + \sum_{k>0} \psi_k(x)$$

for all x. Thus we can rewrite $f * \widehat{\mathrm{d}\omega}$ as

$$f * \widehat{\mathrm{d}\omega} = f * (\phi\widehat{\mathrm{d}\omega}) + \sum_{k>0} f * (\psi_k \widehat{\mathrm{d}\omega}),$$

and, by the triangle inequality

$$\|f * \widehat{\mathrm{d}\omega}\|_{L^{p'}} \leqslant \|f * (\phi\widehat{\mathrm{d}\omega})\|_{L^{p'}} + \sum_{k>0} \|f * (\psi_k \widehat{\mathrm{d}\omega})\|_{L^{p'}}.$$

So it will suffice to bound the right side of this inequality by $\|f\|_{L^p}$.

Now, since $\mathrm{d}\omega$ is finite with compact support $\widehat{\mathrm{d}\omega}$ is a smooth function and so $\phi\widehat{\mathrm{d}\omega}$ will also be a smooth function with compact support. So we have the necessary control via Young's inequality:

$$\|f * \phi(\widehat{\mathrm{d}\omega})\|_{L^{p'}} \leqslant \|\phi\widehat{\mathrm{d}\omega}\|_{L^r} \|f\|_{L^p}$$

where

$$\frac{1}{p'} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Now consider the terms in the summation. The strategy employed for these is to find (L^1, L^∞) and (L^2, L^2) estimates and then interpolate between them.

From Proposition 2.3 we have

$$|\widehat{\mathrm{d}\omega}| \lesssim |x|^{-\frac{n-1}{2}},$$

and since each ψ_k is supported on the annulus $|x| \approx 2^k$ we have

$$\|\psi_k \widehat{\mathrm{d}\omega}\|_{L^\infty} \lesssim 2^{-\frac{(n-1)k}{2}},$$

and so by a trivial application of Young's inequality we have

$$\|f * (\psi_k \widehat{\mathrm{d}\omega})\|_{L^\infty} \lesssim 2^{-\frac{(n-1)k}{2}} \|f\|_{L^1}.$$

Which is the (L^1, L^∞) estimate.

The (L^2, L^2) estimate we will show is

$$\|f * (\psi_k \widehat{\mathrm{d}\omega})\|_{L^2} \lesssim 2^k \|f\|_{L^2}. \tag{2.12}$$

To show this we start with a simple property of convolution kernels, K:

$$\|f * K\|_{L^2} = \|\widehat{f * K}\|_{L^2} = \|\widehat{f}\widehat{K}\|_{L^2} \leqslant \|\widehat{K}\|_{L^\infty} \|\widehat{f}\|_{L^2} = \|\widehat{K}\|_{L^\infty} \|f\|_{L^2},$$

where we have used Plancherel's theorem in the first and final steps. So showing (2.12) is equivalent to showing

$$\|\widehat{\psi_k \widehat{\mathrm{d}\omega}}\|_{L^\infty} \lesssim 2^k,$$

$$\|\widehat{\psi_k} * \mathrm{d}\omega\|_{L^\infty} \lesssim 2^k,$$

(by elementary properties of the inverse Fourier transform and convolution), or

$$|\widehat{\psi_k} * \mathrm{d}\omega(x)| \lesssim 2^k$$

for all x.

From the definition of the ψ_k we have

$$\psi_k(x) = \psi_o(2^{-k}x),$$

and so we have (again by elementary properties of the Fourier transform)

$$\widehat{\psi_k} = 2^{nk}\widehat{\psi_o}(2^k x).$$

Since ψ_o is a Schwartz function, $\widehat{\psi_o}$ is also, and so we must have

$$|\widehat{\psi_k}(x)| \lesssim \frac{2^{nk}}{(1+2^k|x|)^N},$$

for all positive integers N. So we are reduced to showing

$$\left|\frac{2^{nk}}{(1+2^k|x|)^N} * \mathrm{d}\omega(x)\right| \lesssim 2^k.$$

The kernel $\dfrac{2^{nk}}{(1+2^k|x|)^N}$ acts to 'blur' the surface measure to a thickness $\sim 2^k$. So this convolution will still have L^1 norm approximately 1 but since it is now supported on an annulus with thickness 2^{-k} it must have size $\sim 2^k$ (see [24]).

We now use Riesz-Thorin interpolation which states [18] that for a linear operator T, if we have

$$\|T(f)\|_{L^{q_o}} \leqslant M_o\|f\|_{L^{p_o}},$$

and

$$\|T(f)\|_{L^{q_1}} \leqslant M_1 \|f\|_{L^{p_1}},$$

then we have

$$\|T(f)\|_{L^q} \leqslant M_o^{1-\theta} M_1^{\theta} \|f\|_{L^p},$$

for all $0 < \theta < 1$, where

$$\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_o} + \frac{\theta}{q_1}.$$

In our case, then, we have

$$q_o = \infty, p_o = 1, q_1 = 2, p_1 = 2, M_o = 2^{-\frac{(n-1)k}{2}} \quad \text{and} \quad M_1 = 2^k.$$

Therefore,

$$\frac{1}{p} = 1 - \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{2} \quad \text{so} \quad q = p'.$$

Also,

$$M_o^{1-\theta} M_1^{\theta} = 2^{\frac{(n-1)k(\theta-1)}{2}} 2^{\theta k} = 2^{k\left(\frac{(n-1)(\frac{2}{p'}-1)}{2} + \frac{2}{p'}\right)}.$$

Since we are dealing with the infinte sum

$$\sum_{k>0} M_o^{1-\theta} M_1^{\theta} \|f\|_{L^p},$$

to ensure the control by $\|f\|_p$ that we want we require the exponent of 2 to be < 0. In other words

$$\frac{(n-1)(\frac{2}{p'}-1)}{2} + \frac{2}{p'} < 0,$$

which simplifies to

$$\frac{1}{p'} < \frac{n-1}{2n+2},$$

or

$$p < \frac{2n+2}{n+3}.$$

Chapter 3

The Kakeya Conjectures

The ideas expressed in this and the following sections are based on those found in [21] and [31]. As mentioned in the introduction, Kakeya's original problem was to do with finding the minimum area required to rotate a unit line segment by 360°. Through Besicovitch's work this led to the Kakeya set conjecture which is concerned with the dimension of a set with a line segment in every direction. We will introduce another conjecture, the Kakeya maximal operator conjecture, which implies the set conjecture, and through which the best progress on the latter has been made. We will then examine the link between the restriction conjecture and the Kakeya maximal operator conjecture, and show that the former implies the latter. Finally, we will look at the progress made on the Kakeya maximal operator conjecture in the 1990s and compare this with that which is implied by the earlier progress made on the restriction conjecture.

3.1 The Kakeya Maximal Operator Conjecture

This conjecture has two (equivalent) forms:

Conjecture 3.1 *(Kakeya Maximal Operator) For $\delta > 0$, $\omega \in S^{n-1}$ and $a \in \mathbb{R}^n$ let $T_\delta^\omega(a)$ denote the tube in $\mathbb{R}^n$, centred at a, oriented in the ω direction, of length 1 in that direction and cross-sectional radius δ.*

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the Kakeya maximal operator is defined as

$$f^*_\delta(\omega) := \sup_{a\in\mathbb{R}^n} \frac{1}{|T^\omega_\delta(a)|}\int_{T^\omega_\delta(a)} |f|,$$

and it is conjectured that

$$\|f^*_\delta\|_{L^p(S^{n-1})} \lesssim \delta^{-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

for all $\varepsilon > 0$ and for $n \leqslant p \leqslant \infty$.

We also have the trivial inequality:

$$\|f^*_\delta\|_{L^\infty} \leqslant \delta^{-(n-1)}\|f\|_{L^1},$$

and interpolating between this and the Kakeya maximal operator conjecture gives us the family of conjectures

$$\|f^*_\delta\|_{L^q} \lesssim \delta^{-(1-\frac{n}{q})(n-1)-\varepsilon}\|f\|_{L^p} \tag{3.3}$$

for $q \geqslant n$ and $\frac{1}{p} \leqslant 1 - \frac{n-1}{q}$ (which also means $p' \leqslant \frac{n}{n-1}$).

Definition 3.4 *The set of orientations $\{\omega\}$ where $\omega \in S^{n-1}$ is said to be δ-separated if $|\omega - \omega'| > \delta$ for all $\omega, \omega' \in \{\omega\}$.*

The dual form of the conjecture is

Conjecture 3.5 *(Kakeya Maximal Operator (Dual Form)) Let $\mathbb{T}$ be any collection of tubes of length 1 and cross-sectional radius δ (and henceforth referring to such tubes as δ-tubes) whose orientations are δ-separated. Then*

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^p} \lesssim \delta^{\frac{n-1}{p}-\varepsilon}(\#\mathbb{T})^{\frac{1}{p}},$$

for

$$\frac{n}{n-1} \leqslant p \leqslant \infty.$$

3.1.1 The $n = 2$ case

The conjecture has been proved in the case $n = 2$ by Córdoba [24]. Noting that

$$\#\mathbb{T} = \frac{1}{\delta^{n-1}} \sum_{T \in \mathbb{T}} |T|,$$

we can rewrite the Kakeya maximal operator conjecture as

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{\frac{n}{n-1}}} \lesssim \delta^{-\varepsilon} \left(\sum_{T \in \mathbb{T}} |T| \right)^{\frac{n-1}{n}},$$

and we have

Theorem 3.6 *(Kakeya Maximal Operator Conjecture with $n = 2$)*

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^2} \lesssim (\log 1/\delta)^{1/2} \left(\sum_{T \in \mathbb{T}} |T| \right)^{1/2}.$$

Proof. Squaring the left hand side:

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^2}^2 = \int \left(\sum_{T \in \mathbb{T}} \chi_T(x) \right)^2 \mathrm{d}x,$$

$$= \int \sum_{T \in \mathbb{T}} \sum_{T' \in \mathbb{T}} \chi_T(x) \chi_{T'}(x) \mathrm{d}x,$$

$$= \sum_{T \in \mathbb{T}} \sum_{T' \in \mathbb{T}} |T \cap T'|.$$

It suffices to prove

$$\sum_{T' \in \mathbb{T}} |T \cap T'| \lesssim (\log 1/\delta) |T|.$$

Now suppose the tubes T and T' have orientations whose angles differ by $\sim 2^{-k}$ for some $\delta \lesssim 2^{-k} \lesssim 1$. Elementary geometry then yields

$$|T \cap T'| \lesssim 2^k \delta |T|.$$

It now suffices to show that

$$\sum_{k=0}^{\log 1/\delta} \sum_{T' \in \mathbb{T}, \angle(T,T') \sim 2^{-k}} 2^k \delta \lesssim \log(1/\delta).$$

However, for each k there are only $O(\delta^{-1}2^{-k})$ tubes T' whose orientations are within $O(2^{-k})$ of that of the tube T. Hence the result follows. □

3.1.2 The Origin of the Bound in the Kakeya Maximal Operator Conjecture

In an analogous way to that by which the Knapp example demonstrated the $p' > \dfrac{n+1}{n-1}$ bound in the restriction conjecture there is a simple example which demonstrates the $p > \dfrac{n}{n-1}$ bound in the Kakeya maximal operator conjecture:

Let $\mathbb{T}$ be a maximal δ-separated set of δ-tubes all centred at the origin. Consider a point $x_0 \in \mathbb{R}^n$. Clearly if $|x_0| \leqslant \delta/2$ then x_0 is in every tube. Now consider only tubes in a particular plane through the origin. For $\delta/2 \leqslant |x_0| \leqslant 1/2$ the angle subtended between the line from the origin to x_0 and the centre line of a tube on whose edge x_0 lies is $\frac{\delta/2}{|x_0|}$. The angle between adjacent tubes is $\delta/2$ so the number of tubes in which x_0 lies is $1/|x_0|$. Then considering the whole family of tubes we can see that the number of tubes x_0 is in is $\frac{1}{|x_0|^{n-1}}$. In other words

$$\sum_{T \in \mathbb{T}} \chi_T(x) \approx \begin{cases} \delta^{-(n-1)} & \text{if } |x| \leqslant \frac{\delta}{2} \\ |x|^{-(n-1)} & \text{if } \frac{\delta}{2} \leqslant |x| \leqslant \frac{1}{2} \end{cases}.$$

Now

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}^{\frac{n}{n-1}} = C + \int_{\delta/2\leqslant|x|\leqslant 1/2}\left(|x|^{-(n-1)}\right)^{\frac{n}{n-1}} \sim \log(1/\delta),$$

and

$$\int|\sum_{T\in\mathbb{T}}\chi_T|^{\frac{n}{n-1}} \approx \int|\sum_{T\in\mathbb{T}}\chi_T|^1 = \delta^{n-1}(\#\mathbb{T}),$$

so we can not have

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{L^p(\mathbb{R}^n)} \lesssim \delta^{\frac{(n-1)}{p}}(\#\mathbb{T})^{\frac{1}{p}}$$

for any $p < \dfrac{n}{n-1}$.

3.2 The Kakeya Set Conjecture

We first need to introduce some definitions:

Definition 3.7 *(Besicovitch Set) A besicovitch set is defined to be a subset of $\mathbb{R}^n$ which has a unit line segment in every direction*

Definition 3.8 *(Minkowski Dimension) A set E in $\mathbb{R}^n$ has Minkowski dimension d if $\lim_{\delta\to 0}\log_\delta|E_\delta| = n-d$, where E_δ is the δ-neighbourhood of E.*

Then

Conjecture 3.9 *(Kakeya Set) All Besicovitch sets have Minkowski dimension n.*

3.2.1 The Kakeya Maximal Operator Conjecture Implies the Kakeya set Conjecture

Starting with

$$\|f_\delta^*\|_{L^p(S^{n-1})} \leqslant C_\varepsilon\delta^{-\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

consider a zero measure Kakeya set E. Let E_δ be the δ-neighbourhood of E and let $f = \chi_{E_\delta}$.

Then $f_\delta^*(\omega) = 1$ for all $\omega \in S^{n-1}$. So that $\|f_\delta^*\|_{L^p(S^{n-1})} \approx 1$, and $\|f\|_{L^p(\mathbb{R}^n)} = |E_\delta|^{\frac{1}{p}}$ so $|E_\delta|^{\frac{1}{p}} \geqslant C_\varepsilon^{-1}\delta^\varepsilon$.

So all Besicovitch sets have Minkowski dimension n.

3.3 The Restriction Conjecture Implies the Kakeya Maximal Operator Conjecture

This exposition follows that found in [24]. Let us assume that the restriction conjecture holds, in other words

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(S^{n-1})}$$

for all $p' > \dfrac{2n}{n-1}$ and $p' \geqslant \dfrac{n+1}{n-1}q$. So we can say

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{\frac{2n}{n-1}+\varepsilon}} \lesssim \|f\|_{L^{\frac{2n}{n-1}}},$$

for all $\varepsilon > 0$. If we localise $\widehat{f\mathrm{d}\omega}$ to a large ball $B(0,R)$ we have

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{p'}(B(0,R))} \lesssim R^{\frac{n}{p'}}\|\widehat{f\mathrm{d}\omega}\|_{L^\infty} \lesssim R^{\frac{n}{p'}}\|f\|_{L^1} \lesssim R^{\frac{n}{q'}}\|f\|_{L^{\frac{2n}{n-1}}},$$

or

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{\frac{2n}{n-1}+\varepsilon}(B(0,R))} \lesssim R^{\frac{n}{q'}}\|f\|_{L^{\frac{2n}{n-1}}}.$$

So by Hölder's inequality we have (see [9])

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{\frac{2n}{n-1}}(B(0,R))} \lesssim R^{\varepsilon}\|f\|_{L^{\frac{2n}{n-1}}}. \tag{3.10}$$

Let us return to the Knapp example: put f_σ to be the characteristic function of a $\frac{1}{\sqrt{R}}$-cap on the sphere, centred at a point σ in S^{n-1}. This has Fourier transform

$$\widehat{f_\sigma \mathrm{d}\omega}(x) = \int_{|\theta-\sigma|<\frac{1}{\sqrt{R}}} e^{2\pi i x\cdot\theta}\mathrm{d}\theta.$$

We now want to identify those points which are contained within a tube which is in the direction of σ. We do this by breaking up x into the component $(x\cdot\sigma)\sigma$ parallel to σ and the component $(x - x\cdot\sigma)\sigma$ perpendicular to sigma. We then define the tube T_σ^0 to be those points where the parallel component has magnitude $< \frac{R}{100}$ and the perpendicular component has magnitude $< \frac{\sqrt{R}}{100}$.

If $x \in T_\sigma^0$ then

$$\begin{aligned}
|x\cdot(\theta-\sigma)| &= |x||\theta-\sigma|\cos\phi,\\
&\leqslant \frac{x_2}{\cos\phi}\frac{1}{\sqrt{R}}\cos\phi,\\
&\leqslant \frac{\sqrt{R}}{100}\frac{1}{\sqrt{R}},\\
&\leqslant \frac{1}{100}.
\end{aligned}$$

Where ϕ is (90° - the angle subtended by x and σ), and x_2 is the component of x perpendicular to σ. So since $|x\cdot(\theta-\sigma)|$ is small we can rewrite

$$\left|\widehat{f_\sigma \mathrm{d}\omega}(x)\right| \sim \left|\int_{|\theta-\sigma|<\frac{1}{\sqrt{R}}} e^{2\pi i x\cdot\sigma}\mathrm{d}\theta\right|$$

and since the integrand does not depend on θ the integral will equal the measure of the domain (up to a constant) which is $\left(\frac{1}{\sqrt{R}}\right)^{n-1} = R^{-\frac{n-1}{2}}$.

The Kakeya maximal operator conjecture only refers to the orientations of a collection of tubes, not their positions. So if a collection of such tubes T_σ^0 is to meet the requirements of that conjecture they will need to be able to be translated arbitrarily. We can satisfy this by multiplying f_σ by a phase in order to translate $\widehat{f_\sigma \mathrm{d}\omega}$ arbitrarily. This means that for any translate T_σ of T_σ^0 we can find a function f_{T_σ} having size 1 on the arbitrary $\frac{1}{\sqrt{R}}$ cap, center σ such that $\widehat{f_{T_\sigma}\mathrm{d}\omega}$ has size $R^{-\frac{n-1}{2}}$ on T_σ.

If we now define $\Omega = \{\sigma\}$ to be a collection of such caps (i.e. a $1/\sqrt{R}$-separated subset of S^{n-1}) we can also define $\mathbb{T}$ to be the collection of $\sqrt{R} \times R$ tubes which have direction σ for all $\sigma \in \Omega$. For each $T \in \mathbb{T}$ let f_T be the characteristic function of a cap multiplied by the necessary phase such that $\widehat{f_T \mathrm{d}\omega}$ has size $R^{-\frac{n-1}{2}}$ on T.

If we substitute the function

$$\widehat{f\mathrm{d}\omega} = \sum_{T\in\mathbb{T}} \varepsilon_T \widehat{f_T \mathrm{d}\omega},$$

where the ε_T are random ± 1s, into Khinchin's inequality

$$\mathbb{E}\left(\left\|\sum_{k=1}^{N} \varepsilon_k g_k\right\|_{L^{p'}}^{p'}\right) \sim \left\|(\sum_{k=1}^{N} |g_k|^2)^{\frac{1}{2}}\right\|_{L^{p'}}^{p'}$$

we get

$$\mathbb{E}\left(\left\|\widehat{f\mathrm{d}\omega}\right\|_{L^{\frac{2n}{n-1}}(B(0,R))}^{\frac{2n}{n-1}}\right) \sim \left\|(\sum_{T\in\mathbb{T}} |\widehat{f_T\mathrm{d}\omega}|^2)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}(B(0,R))}^{\frac{2n}{n-1}}.$$

With the above estimate on $\widehat{f_T\mathrm{d}\omega}$ this becomes

$$\mathbb{E}\left(\left\|\widehat{f\mathrm{d}\omega}\right\|_{L^{\frac{2n}{n-1}}(B(0,R))}^{\frac{2n}{n-1}}\right) \gtrsim \left\|R^{-\frac{n-1}{2}}(\sum_{T\in\mathbb{T}} \chi_T^2)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}}^{\frac{2n}{n-1}},$$

and on substituting into (3.10) (noting that $f = 1$ on a $\#\mathbb{T}$ of caps of size $R^{-\frac{n-1}{2}}$ so $\|f\|_{L^{\frac{2n}{n-1}}} \sim \left(R^{-\frac{n-1}{2}}\#\mathbb{T}\right)^{\frac{n-1}{2n}}$) we have

$$\left\|R^{-\frac{n-1}{2}}(\sum_{T\in\mathbb{T}} \chi_T^2)^{\frac{1}{2}}\right\|_{L^{\frac{2n}{n-1}}}^{\frac{2n}{n-1}} \lesssim R^{\varepsilon} R^{-\frac{n-1}{2}} \#\mathbb{T}, \tag{3.11}$$

$$R^{-n}\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} \lesssim R^{\varepsilon}R^{-\frac{n-1}{2}}\#\mathbb{T},$$

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} \lesssim R^{\varepsilon}R^{\frac{n+1}{2}}\#\mathbb{T},$$

(Note that each tube has volume $|T| \sim \left(\sqrt{R}^{n-1}\right) R = R^{\frac{n+1}{2}}$),

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} \lesssim R^{\varepsilon}\sum_{T\in\mathbb{T}}|T|,$$

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}} \lesssim R^{\varepsilon}\left(\sum_{T\in\mathbb{T}}|T|\right)^{\frac{n-1}{n}}.$$

If we set the tubes T to have length 1 and thickness δ we have

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}} \lesssim \delta^{-\varepsilon}(\delta^{n-1}\#\mathbb{T})^{\frac{n-1}{n}},$$

and so arrive at the end-point of the Kakeya maximal operator conjecture

$$\left\|\sum_{T\in\mathbb{T}}\chi_T\right\|_{L^{\frac{n}{n-1}}} \lesssim \delta^{\frac{(n-1)^2}{n}-\varepsilon}(\#\mathbb{T})^{\frac{n-1}{n}}. \tag{3.12}$$

3.4 Progress on the Kakeya Maximal Operator Conjecture

We can show [24] via what is known as factorisation theory and which utilises the rotational symmetry of the sphere, that (3.12) is equivalent to

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{L^{\frac{n}{n-1}}} \lesssim \delta^{-\varepsilon}.$$

Also, since the tubes are δ-separated in direction there can only be δ^{1-n} of them and so this value is the maximum number of tubes there can be overlapping at any point, in other words

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{L^\infty} \lesssim \delta^{1-n}.$$

Interpolating between these two yields the family of conjectures

$$\|\sum_{T\in\mathbb{T}}\chi_T\|_{L^p} \lesssim \delta^{\frac{n}{p}-(n-1)-\varepsilon}. \tag{3.13}$$

The goal of work on the Kakeya conjectures is to prove this for as low a p as possible and, ultimately, for $p = \frac{n}{n-1}$.

Since the restriction conjecture implies the Kakeya maximal operator conjecture it is reasonable to assume that the partial results obtained for the former will yield information on the latter and hence the Kakeya set conjecture. We will ascertain just how much information in the next subsection. In the 1990s Bourgain in [6] and Wolff in [30] found ways to attack the Kakeya conjectures directly. These involved the geometric considerations of a collection of, possibly overlapping, δ-tubes. Bourgain was able to show that we have (3.13) for $p = \frac{n+1}{n-1}$ and Wolff for $p = \frac{n+2}{n}$. We give Bourgain's argument below but we postpone Wolff's until we have discussed bilinear estimates in the next chapter.

3.4.1 Bourgain's Bush Argument

This subsection is motivated by [1]. Let

$$E_\mu = \{x : \sum_{T\in\mathbb{T}}\chi_T(x) > \mu\}.$$

Observe that

$$\int_{E_\mu} \sum_{T\in\mathbb{T}} \chi_T(x) > \mu|E_\mu| \Leftrightarrow \sum_{T\in\mathbb{T}} |T\cap E_\mu| > \mu|E_\mu| \Leftrightarrow \frac{1}{\#\mathbb{T}} \sum_{T\in\mathbb{T}} |T\cap E_\mu| > \frac{\mu|E_\mu|}{\#\mathbb{T}}.$$

Now let us discard those tubes which contain less than a certain amount of E_μ: define

$$\widetilde{\mathbb{T}} = \{T\in\mathbb{T} : |T\cap E_\mu| > \frac{\mu|E_\mu|}{10^n\#\mathbb{T}}\}.$$

Now we can observe that

$$\int_{E_\mu} \sum_{T\in\widetilde{\mathbb{T}}} \chi_T = \sum_{T\in\widetilde{\mathbb{T}}} |T\cap E_\mu| = \sum_{T\in\mathbb{T}} |T\cap E_\mu| - \sum_{T\in\mathbb{T}\setminus\widetilde{\mathbb{T}}} |T\cap E_\mu| \geqslant \left(\frac{10^n-1}{10^n}\mu|E_\mu|\right) \gtrsim \mu|E_\mu|.$$

So

$$\frac{1}{|E_\mu|} \int_{E_\mu} \sum_{T\in\widetilde{\mathbb{T}}} \chi_T \gtrsim \mu.$$

This means that, on the average over E_μ, there are at least $\gtrsim \mu$ tubes overlapping so we can identify a point, x_o, in E_μ which must be contained in at least $\gtrsim \mu$ tubes. If we let

$$\widetilde{\mathbb{T}}_{x_o} = \{T\in\widetilde{\mathbb{T}} : x_o\in T\},$$

we know $\#\widetilde{\mathbb{T}}_{x_o} \gtrsim \mu$. $\widetilde{\mathbb{T}}_{x_o}$ is the 'bush' after which this argument is named.

We are looking to bound $|E_\mu|$ from below by saying that there must be at least that much E_μ which is contained in a set of tubes whose regions containing the E_μ are disjoint. $\widetilde{\mathbb{T}}_{x_o}$ is a δ-separated set of tubes so to ensure the disjointness that we require we must take a subset of $\widetilde{\mathbb{T}}_{x_o}$. Let us define $\widetilde{\widetilde{\mathbb{T}}}_{x_o}$ to be a $\delta/(\mu|E_\mu|)$-separated subset of $\widetilde{\mathbb{T}}_{x_o}$, having cardinality $\gtrsim \mu(\mu|E_\mu|)^{n-1}$. If we now choose an r such that, for a ball $B(x_o,r)$,

$$|T\cap B(x_o,r)| \lesssim \frac{\mu|E_\mu|}{\#\mathbb{T}},$$

for all $T \in \widetilde{\mathbb{T}}_{x_o}$, then for such an r we have

$$|T \cap E_\mu \cap B(x_o, r)^c| \gtrsim \frac{\mu |E_\mu|}{\#\mathbb{T}},$$

for all $T \in \widetilde{\mathbb{T}}_{x_o}$. If we now restrict our attention to $\widehat{\widetilde{\mathbb{T}}}_{x_o}$ we see that the sets on the left of this inequality are disjoint. Each of these sets has measure $\gtrsim \frac{\mu|E_\mu|}{\#\mathbb{T}}$ and we have $\gtrsim \mu(\mu|E_\mu|)^{n-1}$ of them, so we can say

$$|E_\mu| \gtrsim \frac{\mu|E_\mu|}{\#\mathbb{T}} \mu(\mu|E_\mu|)^{n-1},$$

or, putting $\#\mathbb{T} = \delta^{-(n-1)}$

$$|E_\mu| \gtrsim \mu^{n+1}|E_\mu|^n \delta^{n-1},$$

$$\delta \gtrsim \mu^{\frac{n+1}{n-1}}|E_\mu|.$$

Define new, dyadic, sets $\widetilde{E}_\mu$:

$$\widetilde{E}_\mu = \{x : \mu \leqslant \sum_{T \in \mathbb{T}} \chi_T(x) < 2\mu\}.$$

Note that $|\widetilde{E}_\mu| \leqslant |E_\mu|$. Now

$$\int \left(\sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n+1}{n-1}} \mathrm{d}x = \sum_\mu \int_{\widetilde{E}_\mu} \left(\sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n+1}{n-1}} \mathrm{d}x,$$

$$\lesssim \sum_\mu \mu^{\frac{n+1}{n-1}} |\widetilde{E}_\mu|.$$

The number of dyadic μ is logarithmic in δ so

$$\int \left(\sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n+1}{n-1}} \mathrm{d}x \lessapprox \mu^{\frac{n+1}{n-1}} |\widetilde{E}_\mu|.$$

Where $A \lessapprox B$ means $A \leqslant C_\varepsilon \delta^{-\varepsilon} B$ for every $\varepsilon > 0$. So we have

$$\int \left(\sum_{T \in \mathbb{T}} \chi_T(x) \right)^{\frac{n+1}{n-1}} \mathrm{d}x \lessapprox \delta,$$

or

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{\frac{n+1}{n-1}}} \lesssim \delta^{\frac{n-1}{n+1} - \varepsilon},$$

i.e. (3.13) for $p = \frac{n+1}{n-1}$.

3.5 The Correspondence between partial restriction and partial Kakeya Estimates

Following a similar argument to that found in section (3.3) but without assigning values to the exponents p', q' enables us to analyse the correspondence between the respective progress made on the restriction and Kakeya set conjectures.

Let $\mathbb{T}$ be a collection of tubes satisfying the assumptions of the Kakeya maximal operator conjecture (i.e. $\mathbb{T}$ is an arbitrary collection of δ-tubes whose orientations are δ-separated). For $T \in \mathbb{T}$ let $\widetilde{T} = \delta^{-2} T$. So $\widetilde{\mathbb{T}} = (\widetilde{T})$ is a collection of $\delta^{-1} \times \delta^{-2}$ tubes.

As in section (3.3), for each $\widetilde{T} \in \widetilde{\mathbb{T}}$ there exists a function $f_{\widetilde{T}}$ on S such that $|\widehat{f_{\widetilde{T}} \mathrm{d}\sigma}| \sim \delta^{n-1}$ on T. After applying Khinchin's inequality and utilising $\|f\|_{L^{q'}} \sim (\delta^{n-1} \# \mathbb{T})^{1/q'}$ we arrive at the analogue of (3.11)

$$\left\| \delta^{n-1} \Big(\sum_{\widetilde{T} \in \widetilde{\mathbb{T}}} \chi^2_{\widetilde{T}} \Big)^{1/2} \right\|^{p'}_{L^{p'}} \lesssim (\delta^{n-1} \# \mathbb{T})^{p'/q'}.$$

Now

$$\int_{\mathbb{R}^n} |\sum_{\widetilde{T} \in \widetilde{\mathbb{T}}} \chi_{\widetilde{T}}(x)|^{p'/2} \mathrm{d}x = \int_{\mathbb{R}^n} |\sum_{T \in \mathbb{T}} \chi_T(\delta^2 x)|^{p'/2} \mathrm{d}x,$$

and

$$x \in \widetilde{T} \Leftrightarrow \delta^2 x \in T$$

so if we let $y = \delta^2 x$ we have

$$\int_{\mathbb{R}^n} |\sum_{T\in\mathbb{T}} \chi_T(\delta^2 x)|^{p'/2} \mathrm{d}x = \int_{\mathbb{R}^n} |\sum_{T\in\mathbb{T}} \chi_T(y)|^{p'/2} \delta^{-2n} \mathrm{d}y = \delta^{-2n} \|\sum_{T\in\mathbb{T}} \chi_T\|_{L^{p'/2}}^{p'/2}.$$

So we have

$$\delta^{(n-1)p'-2n} \left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^{p'/2}}^{p'/2} \lesssim \delta^{\frac{(n-1)p'}{q'}} (\#\mathbb{T})^{p'/q'},$$

$$\delta^{2(n-1)-\frac{4n}{p'}} \left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^{p'/2}} \lesssim \delta^{\frac{2(n-1)}{q'}} (\#\mathbb{T})^{2/q'},$$

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{L^{p'/2}} \lesssim \delta^{2(n-1)(\frac{1}{q'}-1)+4n/p'} (\#\mathbb{T})^{2/q'}. \tag{3.14}$$

Now, let E be a Besicovitch set and E_δ be the δ-neighbourhood of E. Then there exists a maximal δ-separated family $\mathbb{T}$ of δ-tubes with $T \subseteq E_\delta$ for all $T \in \mathbb{T}$. In particular $\bigcup_{T\in\mathbb{T}} \subseteq E_\delta$. We then observe, by Hölder's inequality,

$$1 \approx \sum_{T\in\mathbb{T}} |T| = \int \sum_{T\in\mathbb{T}} \chi_T = \int_{E_\delta} (\sum_{T\in\mathbb{T}} \chi_T) \chi_{E_\delta} \leqslant \|\sum_{T\in\mathbb{T}} \chi_T\|_{L^{p'}} |E_\delta|^{\frac{1}{p}}.$$

Where the first inequality follows since $|T| = \delta^{n-1}$ and $\mathbb{T} = \delta^{-(n-1)}$ and the volume of any overlap of two tubes will be $\mathrm{O}(\delta^n)$. If we now substitute (3.14) (noting that the conjugate exponent of $p'/2$ is $(p'-2)/p'$, and that $\#\mathbb{T} = \delta^{-(n-1)}$) we obtain

$$1 \leqslant \delta^{2(n-1)(\frac{1}{q'}-1)+\frac{4n}{p'}-\frac{2(n-1)}{q'}} |E_\delta|^{\frac{p'}{p'-2}},$$

$$|E_\delta|^{\frac{p'}{p'-2}} \geqslant \delta^{2(n-1)-\frac{4n}{p'}},$$

and, after some algebra,

$$|E_\delta| \geqslant \delta^{2n-\frac{2p'}{p'-2}}.$$

If we express this in the form

$$|E_\delta| \geqslant \delta^{n-(\frac{2p'}{p'-2}-n)}$$

we can see that the Minkowski dimension of E is given by $\dim(E) = \frac{2p'}{p'-2} - n$.

We can use this relationship to deduce what known restriction estimates tell us about the dimension of Besicovitch sets and also to calculate what restriction estimates we would require to recover known bounds on the dimension of such sets.

If we substitute the Stein-Tomas exponent $p' = \frac{2(n+1)}{n-1}$ we have

$$\frac{2p'}{p'-1} - n = \frac{\frac{4(n+1)}{n-1}}{\frac{2(n+1)}{n-1} - 2} - n = \frac{4(n+1)}{2(n+1) - 2(n-1)} - n = \frac{4(n+1)}{4} - n = 1.$$

In other words, the Minkowski dimension of a Besicovitch set is at least 1, so the Stein-Tomas restriction estimate tells us nothing about the Kakeya Set conjecture.

Bourgain's 'bush' argument tell us that $\dim(E) \geqslant \frac{n+1}{2}$. To recover this bound from a restriction theorem would require a p' exponent such that

$$\frac{n+1}{2} = \frac{2p'}{p'-2} - n,$$

$$p' = \frac{2(3n+1)}{3(n-1)}.$$

Similarly for Wolff's 'hairbrush' argument $\dim(E) \geqslant \frac{n+2}{2}$, which would require a p' exponent such that

$$\frac{n+2}{2} = \frac{2p'}{p'-2} - n,$$

$$p' = \frac{2(3n+2)}{3n-2}.$$

Since both of these values of p' are less than the best known bound in the restriction conjecture (namely $p' \geqslant \frac{2(n+2)}{n}$) for all $n \geqslant 3$ we see that Bourgain's and Wolff's 'direct' geometrical methods have made more progress in attacking the set conjecture than is implied by known restriction estimates.

Chapter 4

Bilinear Estimates

The concept of using bilinear estimates in the restriction problem dates back to the 1970s but the modern approach is based on the work of Bourgain in the 1990s in [6], [7], [8], [9] and [10]. Until very recently (the 2010 paper of Bourgain and Guth [12]) it was via bilinear estimates that the best linear estimates were obtained. Our approach is based largely on [27]: we first consider the bilinear analogue to the restriction conjecture and justify its best possible exponents; we then use the approach from [12], as interpreted in [3] to show that the bilinear conjecture implies the linear conjecture. Finally we look at bilinear analogue of the Kakeya maximal operator conjecture, which is the best way to introduce the best known Kakeya estimate derived from Wolff's 'hairbrush' argument.

4.1 Bilinear Restriction Estimates

The material in this section is based on [24]. The origins of the study of bilinear restriction estimates were 'L^4' or bi-orthogonality theory investigated in such places as [13], [15], and [17]. The basic idea is to rewrite, by an application of Plancherel's theorem, an expression such as $\|\widehat{f\mathrm{d}\omega}\|_{L^4}$ as

$$\|\widehat{f\mathrm{d}\omega}\|_{L^4} = \|\widehat{f\mathrm{d}\omega}\widehat{f\mathrm{d}\omega}\|_{L^2}^{\frac{1}{2}} = \|f\mathrm{d}\omega * f\mathrm{d}\omega\|_{L^2}^{\frac{1}{2}}.$$

This enables us to rewrite the extension estimate

$$\|\widehat{f\mathrm{d}\omega}\|_{L^4} \lesssim \|f\|_{L^{q'}(S;\mathrm{d}\omega)},$$

as

$$\|f\mathrm{d}\omega * f\mathrm{d}\omega\|_{L^2}^{\frac{1}{2}} \lesssim \|f\|_{L^{q'}(S;\mathrm{d}\omega)}.$$

Notice that there is no Fourier transform in this estimate, rendering the determination of its truth accessible to more direct methods than those used thus far. This elimination of the Fourier transform is only possible when we are dealing with an even integer exponent p in $\|\widehat{f\mathrm{d}\omega}\|_{L^p}$, but this approach can be applied, to a lesser extent, for all values of p:

$$\|\widehat{f\mathrm{d}\omega}\|_{L^{p'}} \lesssim \|f\|_{L^{q'}(S;\mathrm{d}\omega)},$$

is equivalent to

$$\|\widehat{f\mathrm{d}\omega}\widehat{f\mathrm{d}\omega}\|_{L^{p'/2}} \lesssim \|f\|_{L^{q'}(S;\mathrm{d}\omega)}\|f\|_{L^{q'}(S;\mathrm{d}\omega)},$$

which is a special case of the bilinear estimate

$$\|\widehat{f_1\mathrm{d}\omega}\widehat{f_2\mathrm{d}\omega}\|_{L^{p'/2}} \lesssim \|f_1\|_{L^{q'}(S;\mathrm{d}\omega)}\|f_2\|_{L^{q'}(S;\mathrm{d}\omega)},$$

which is itself a special case of

$$\|\widehat{f_1\mathrm{d}\omega}\widehat{f_2\mathrm{d}\omega}\|_{L^{p'/2}} \lesssim \|f_1\|_{L^{q'}(S_1;\mathrm{d}\omega)}\|f_2\|_{L^{q'}(S_2;\mathrm{d}\omega)}, \tag{4.1}$$

i.e. an extension estimate which is true for arbitrary pairs of smooth, compact hypersurfaces S_1, S_2 and all smooth f_1, f_2 supported on S_1, S_2, respectively. Let us denote (4.1) by

$$R^*_{S_1,S_2}(q' \times q' \to p'/2).$$

So we can see that linear estimates are just special cases of bilinear ones. For every linear estimate $R_S^*(q' \to p')$ there is a corresponding bilinear estimate $R_{S,S}^*(q' \times q' \to p'/2)$ but the converse is not true. The following example, found in [24] illustrates this.

Example

First let us define the concept of transversality:

Definition 4.2 *We say that the k-tuple $S_1, \dots, S_k$ is transversal if there exists a constant $c > 0$ such that*

$$|v_1 \times \cdots \times v_k| \geqslant c,$$

for all choices of unit normal vectors $v_1, \dots, v_k$ to $S_1, \dots, S_k$ respectively.

Let $S_1 := \{(\xi_1, 0) : \xi_1 \in \mathbb{R}\}$ and $S_2 := \{(0, \xi_2) : \xi_2 \in \mathbb{R}\}$ denote the x and y axes respectively in $\mathbb{R}^2$. Then we have $\widehat{f_1 \mathrm{d}\omega_1}(x,y) = \widehat{f}_1(x)$ and $\widehat{f_2 \mathrm{d}\omega_2}(x,y) = \widehat{f}_2(y)$ and so we only have $R_{S_1}^*(q' \to p')$ and $R_{S_2}^*(q' \to p')$ if $p' = \infty$, since $\widehat{f_1 \mathrm{d}\omega_1}$ does not decay in the y-direction and $\widehat{f_2 \mathrm{d}\omega_2}$ does not decay in the x-direction.

However, since

$$\widehat{f_1 \mathrm{d}\omega_1}\widehat{f_2 \mathrm{d}\omega_2}(x,y) = \widehat{f}_1(x)\widehat{f}_2(y),$$

and

$$\|\widehat{f}_1(x)\widehat{f}_2(y)\|_{L^2(\mathbb{R}^2)} = \|\widehat{f}_1(x)\|_{L^2(\mathbb{R}^2)}\|\widehat{f}_2(y)\|_{L^2(\mathbb{R}^2)},$$

by the 1-dimensional Plancherel theorem we have

$$\|\widehat{f}_1(x)\|_{L^2(\mathbb{R}^2)}\|\widehat{f}_2(y)\|_{L^2(\mathbb{R}^2)} \lesssim \|f_1\|_{L^2(S_1)}\|f_2\|_{L^2(S_2)},$$

and so we *do* have the bilinear extension estimate $R_{S_1,S_2}^*(2 \times 2 \to 2)$.

Note that we do *not* have $R^*_{S_1,S_1}(2 \times 2 \to 2)$ or $R^*_{S_2,S_2}(2 \times 2 \to 2)$. So the bilinear extension estimate relies on the *transversality* of S_1 and S_2.

Note also that we do not have the linear estimate $R^*_S(2 \to 4)$ since we must have $p' \geqslant \frac{n+1}{n-1}q$ so in this case we need $p' \geqslant 3q$ but $p' = 4$ and $q = 2$.

4.1.1 Knapp Example in the Bilinear Setting

The reason why a larger range of exponents p, q is permissable in the bilinear setting can be seen by applying the Knapp example in the bilinear setting as was done in the linear case. If we try to directly replicate the Knapp example in the linear case, i.e. by putting f_1 and f_2 equal to the respective characteristic functions of spherical caps S_1, S_2 of area $\lambda^{-(n-1)}$ then $\widehat{f_1 \mathrm{d}\omega}$ and $\widehat{f_2 \mathrm{d}\omega}$ both have size $\sim \lambda^{-(n-1)}$ on tubes T_1, T_2 of volume $\sim \lambda^{n+1}$. However, since S_1 and S_2 are transversal the intersection of T_1 and T_2 will be $\sim \lambda^n$. So

$$\|\widehat{f_1 \mathrm{d}\omega}\widehat{f_2 \mathrm{d}\omega}\|_{L^{p'/2}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{q'}(S_1)}\|f_2\|_{L^{q'}(S_2)},$$

becomes

$$(\lambda^{-(n-1)p'+n})^{2/p'} \lesssim \lambda^{-2\frac{n-1}{q'}},$$

and on letting $\lambda \to \infty$ we have

$$-(n-1) + \frac{n}{p'} \leqslant -\frac{n-1}{q'},$$

which, after some algebra, gives

$$\frac{n}{p'} \leqslant \frac{n-1}{q}.$$

However, by modifying the Knapp example (as detailed in theorem 4.7) we can obtain

Conjecture 4.3 *(Bilinear Restriction) If S_1, S_2 are transversal and have non-vanishing Gaussian curvature,*

$$p \leqslant \frac{2n}{n+1}; \tag{4.4}$$

$$\frac{n+2}{p'} + \frac{n}{q'} \leqslant n; \tag{4.5}$$

$$\frac{n+2}{p'} + \frac{n-2}{q'} \leqslant n-1 \tag{4.6}$$

then $R^*_{S_1,S_2}(q' \times q' \to \frac{p'}{2})$ holds.

Theorem 4.7 *The above exponents are the best possible.*

Proof. If we take f_1 to be the characteristic function of S_1 and f_2 to be the characteristic function of S_2 but multiplied by some phase (which we will ascertain). The ideas that resulted in proposition 2.3 can be extended to all hypersurfaces, S, of non-zero Gaussian curvature (see [23]). In particular we have

$$|\widehat{\psi \mathrm{d}\omega(x)}| \lesssim |x|^{-\frac{n-1}{2}}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ whose support intersects S in a compact subset of S. So for any $\lambda \gg 1$ we can find a cube C such that $|\widehat{f_1 \mathrm{d}\omega}| \sim \lambda^{-\frac{n-1}{2}}$ on C. We can also choose the phase which we multiply f_2 by to ensure $|\widehat{f_2 \mathrm{d}\omega}| \sim \lambda^{-\frac{n-1}{2}}$ on C also. If we substitute these estimates into (4.1) we obtain

$$\lambda^{-\frac{n-1}{2}} \lambda^{-\frac{n-1}{2}} |C|^{\frac{2}{p'}} \lesssim 1.$$

Now if we substitute $|C| \sim \lambda^n$ and let $\lambda \to \infty$ we have

$$\frac{2n}{p'} \leqslant n-1,$$

from which (4.4) follows.

To prove (4.5) we use the 'squashed caps' example from [27]: if we now take as our caps tubes of dimensions $1/R \times 1/R \times 1/\sqrt{R} \times \cdots \times 1/\sqrt{R}$ (i.e. the usual $1/\sqrt{R}$ cap but with one dimension 'squashed' down to $1/R$) with their long sides parallel and, as usual, f_1 and f_2 to be the characteristic functions of these caps, then we have $|\widehat{f_1 \mathrm{d}\omega}| \sim |\widehat{f_2 \mathrm{d}\omega}| \sim R^{-n/2}$ on a $R \times R \times \sqrt{R} \times \cdots \times \sqrt{R}$ box in $\mathbb{R}^n$. Again substituting these estimates into (4.1) yields

$$R^{-\frac{n}{2}} R^{-\frac{n}{2}} (R^2 (R^{\frac{1}{2}})^{n-2})^{\frac{2}{p'}} \lesssim (R^{-1}(R^{-1/2})^{n-2})^{\frac{2}{q'}},$$

$$R^{-n} R^{\frac{n+2}{p'}} \lesssim R^{-\frac{n}{q'}},$$

which, on taking $R \to \infty$ gives us (4.5).

For (4.6) we use the 'stretched caps' example, also found in [27]: if we now take f_1 and f_2 to be the characteristic functions (possibly multiplied by a phase) of

$$S_i \cap (\mathbb{R}^2 \times B_{n-2}(0, \frac{1}{\sqrt{R}})), \qquad i = 1, 2,$$

respectively, where $B_{n-2}(0, \frac{1}{\sqrt{R}})$ is the ball in $\mathbb{R}^{n-2}$, centre 0, radius $1/\sqrt{R}$, then, by stationary phase estimates again, we have

$$\widehat{f_1 \mathrm{d}\omega} \sim R^{-\frac{n-2}{2}} |x|^{-\frac{1}{2}}$$

on a large portion of the slab

$$\mathbb{R}^2 \times B_{n-2}(0, \frac{\sqrt{R}}{C}),$$

for some constant C (similarly for $\widehat{f_2 \mathrm{d}\omega}$). So by choosing a phase to translate $\widehat{f_1 \mathrm{d}\omega}$ and $\widehat{f_2 \mathrm{d}\omega}$ sufficiently we have

$$\widehat{f_1 \mathrm{d}\omega} \sim \widehat{f_2 \mathrm{d}\omega} \sim R^{-\frac{n-2}{2}} R^{-\frac{1}{2}} = R^{-\frac{n-1}{2}},$$

on

$$B_2(0, \frac{R}{C}) \times B_{n-2}(0, \frac{\sqrt{R}}{C}).$$

Substituting these estimates into (4.1) we get

$$R^{-\frac{n-1}{2}} R^{-\frac{n-1}{2}} (R^{\frac{n+2}{2}})^{\frac{2}{p'}} \lesssim R^{-\frac{n-2}{2q}} R^{-\frac{n-2}{2q}},$$

and on taking $R \to \infty$ we get

$$-(n-1) + \frac{n+2}{p'} \leqslant -\frac{n-2}{q},$$

which is (4.6). □

4.1.2 The Bilinear Restriction Conjecture Implies the Linear Restriction Conjecture

Definition 4.8 *We define $\Phi : Q \to \mathbb{R}$, where Q is the cube $[-1, 1]^{n-1}$, to be an* elliptic phase function, *if, for fixed $n \geqslant 2$ and $A > 0$, $\|\partial^\alpha \Phi\|_{L^\infty} \leqslant A$ for all $0 \leqslant |\alpha| \leqslant N$, where N is a large constant, $\Phi(0) = \nabla\Phi(0) = 0$, and, for all $x \in Q$ the eigenvalues of the Hessian matrix of Φ at $x, \partial_i\partial_j(x)$, are contained in the interval $[1 - \varepsilon_0, 1 + \varepsilon_0]$, for a constant $0 < \varepsilon_0 \ll 1$.*

Any smooth, compact, convex surface of non-vanishing curvature (in particular the unit sphere) can be comprised of finitely many graphs of elliptic phase functions (after an affine transformation) [27].

The implication of the linear conjecture by the bilinear one involves a technical complication that requires a version of the bilinear conjecture that behaves well under scaling of the surfaces over which the Fourier transform is taken (see [27] and [2] for details).

Hence we also define the operator $\mathcal{R}^* : L^1(Q) \to L^\infty(\mathbb{R}^n)$ by

$$\mathcal{R}^* f(\underline{x}, x_n) = \int_Q e^{-2\pi i(\underline{x}\cdot y + x_n \Phi(y))} f(y)\mathrm{d}y,$$

in other words, an extension operator associated with the surface $\{(y, \Phi(y) : y \in Q)\}$.

Since the conditions for $R^*_{S_1,S_2}(q' \times q' \to \frac{p'}{2})$ to hold are weaker than those for $R^*_S(p' \to q')$ to hold we can not expect the former to unconditionally imply the latter. However, the following theorem *does* enable us to infer linear estimates from bilinear ones. The proof we give is based on that found in [3], which is an interpretation of the method found in [12].

Theorem 4.9 *(Tao-Vargas-Vega 1998)*

Suppose that S is as in definition 4.8 and that S_1 and S_2 are transversal subsets of S. If $p < \dfrac{2n}{n+1}$ and $p' \geqslant \dfrac{n+1}{n-1}q$ and the conjectured bilinear inequality

$$\|\mathcal{R}^* f_1 \mathcal{R}^* f_2\|_{L^{\widetilde{p'}/2}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{\widetilde{q'}}(S_1)} \|f_2\|_{L^{\widetilde{q'}}(S_2)},$$

holds for all $(\widetilde{p'}, \widetilde{q'})$ in a neigbourhood of (p', q'), then the conjectured linear inequality

$$\|\mathcal{R}^* f\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}(\mathrm{d}\omega)},$$

holds for (p', q').

To prove this theorem we need the following proposition of Bourgain and Guth:

Proposition 4.10 *Let $\{S_\alpha\}$ be a partition of S by caps of diameter approximately $1/K$ (where K is a large parameter), so $F = \sum_\alpha F_\alpha$ where $F_\alpha = F \cdot \chi_{S_\alpha}$. Then*

$$|\widehat{F\mathrm{d}\omega}(\xi)|^{p'} \lesssim K^{2(n-1)p'} \sum_{dist(S_{\alpha_1}, S_{\alpha_2}) \gtrsim 1/K} |\widehat{F_{\alpha_1}\mathrm{d}\omega}(\xi)\widehat{F_{\alpha_2}\mathrm{d}\omega}(\xi)|^{\frac{p'}{2}} + \sum_\alpha |\widehat{F_\alpha \mathrm{d}\omega}(\xi)|^{p'}.$$

Proof. (Motivated by [3]). For a given $\xi \in \mathbb{R}^n$ either

(1) there exist α_1, α_2 with $\mathrm{dist}(S_{\alpha_1}, S_{\alpha_2}) \gtrsim 1/K$ such that

$$|\widehat{F_{\alpha_1}\mathrm{d}\omega}(\xi)|, |\widehat{F_{\alpha_2}\mathrm{d}\omega}(\xi)| \geqslant K^{-(n-1)}\max_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|,$$

or

(2) there exists α_0 such that whenever $\mathrm{dist}(S_{\alpha_0}, S_\alpha) \gtrsim 1/K$

$$|\widehat{F_\alpha \mathrm{d}\omega}(\xi)| < K^{-(n-1)}\max_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|.$$

If (1) then

$$\begin{aligned}
|\widehat{F\mathrm{d}\omega}(\xi)| &\leqslant \sum_\alpha |\widehat{F_\alpha \mathrm{d}\omega}(\xi)| \lesssim K^{n-1}\max_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|,\\
&\leqslant K^{2(n-1)}|\widehat{F_{\alpha_1(\xi)}\mathrm{d}\omega}(\xi)|^{1/2}|\widehat{F_{\alpha_2(\xi)}\mathrm{d}\omega}(\xi)|^{1/2},\\
&\leqslant K^{2(n-1)}\left(\sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K}|\widehat{F_{\alpha_1}\mathrm{d}\omega}(\xi)\widehat{F_{\alpha_2}\mathrm{d}\omega}(\xi)|^{\frac{p'}{2}}\right)^{\frac{1}{p'}}.
\end{aligned}$$

If (2) then

$$\begin{aligned}
|\widehat{F\mathrm{d}\omega}(\xi)| &\leqslant \sum_\alpha |\widehat{F_\alpha \mathrm{d}\omega}(\xi)|,\\
&\leqslant \sum_{\alpha:\mathrm{dist}(S_\alpha,S_{\alpha_0})\lesssim 1/K}|\widehat{F_\alpha \mathrm{d}\omega}(\xi)| + \sum_{\alpha:\mathrm{dist}(S_\alpha,S_{\alpha_0})\gtrsim 1/K}|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|,\\
&\lesssim \max_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)| + K^{n-1}K^{-(n-1)}\max_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|,\\
&\lesssim \left(\sum_\alpha |\widehat{F_\alpha \mathrm{d}\omega}(\xi)|^{p'}\right)^{1/p'}.
\end{aligned}$$

So we can conclude that

$$|\widehat{F\mathrm{d}\omega}(\xi)|^{p'} \lesssim K^{2(n-1)p'}\sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K}|\widehat{F_{\alpha_1}\mathrm{d}\omega}(\xi)\widehat{F_{\alpha_2}\mathrm{d}\omega}(\xi)|^{\frac{p'}{2}} + \sum_\alpha|\widehat{F_\alpha \mathrm{d}\omega}(\xi)|^{p'},$$

□

We are now in a position to prove theorem 4.9. We prove it for $p' = q' > \frac{2n}{n-1}$ which implies the linear restriction conjecture for the full range of conjectured exponents.

Proof. (theorem 4.9) Integrating the inequality from Proposition 4.10 in $\xi \in \mathbb{R}^n$ we have

$$\|\widehat{F\mathrm{d}\omega}\|_{L^{p'}}^{p'} \lesssim K^{2(n-1)p'} \sum_{\mathrm{dist}(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K} \|\widehat{F_{\alpha_1}\mathrm{d}\omega}\widehat{F_{\alpha_2}\mathrm{d}\omega}\|_{L^{\frac{p'}{2}}}^{\frac{p'}{2}} + \sum_{\alpha} \|\widehat{F_\alpha \mathrm{d}\omega}\|_{L^{p'}}^{p'}. \tag{4.11}$$

Let $C = C(R)$ denote the best constant in the inequality

$$\|\widehat{F\mathrm{d}\omega}\|_{L^{p'}(B(0,R))} \leqslant C\|F\|_{L^{p'}(\mathrm{d}\omega)}$$

for all $R \gg 1$ and all $F \in L^{q'}(\mathrm{d}\omega)$. The purpose of using R here is to ensure the finiteness of C. Our aim is to show that $C < \infty$ uniformly in R.

This inequality scales as ([4] and [12])

$$\|\mathcal{R}^* F_\alpha\|_{L^{p'}(B(0,R))} \leqslant CK^{\frac{2n}{p'}-(n-1)}\|F_\alpha\|_{L^{p'}(\mathrm{d}\omega)}.$$

Note that we are now utilising the properties of the operator $\mathcal{R}^*$, introduced at the start of this subsection, which are important in overcoming some technical difficulties associated with this scaling (see [2], [4] and [12]).

With this inequality and $F = \sum_\alpha F_\alpha$ we can rewrite (4.11) (now in terms of $\mathcal{R}^*F$):

$$\|\mathcal{R}^*F\|_{L^{p'}}^{p'} \leqslant cK^{2(n-1)p'} \sum_{dist(S_{\alpha_1},S_{\alpha_2})\gtrsim 1/K} \|\widehat{F_{\alpha_1}\mathrm{d}\omega}\widehat{F_{\alpha_2}\mathrm{d}\omega}\|_{L^{p'/2}}^{p'/2} + cCK^{\frac{2n}{p'}-(n-1)}\|F\|_{L^{p'}}^{p'}.$$

Note that the power of K in the second term on the right, $\frac{2n}{p'} - (n-1)$ is negative. So we can set K to make the second term equal to an arbitrarily small constant, a. Now by the hypothesis of theorem 4.9 it follows that

$$K^{2(n-1)p'} \sum_{dist(S_{\alpha_1}, S_{\alpha_2}) \gtrsim 1/K} \| \widehat{F_{\alpha_1} \mathrm{d}\omega} \widehat{F_{\alpha_2} \mathrm{d}\omega} \|_{L^{p'/2}}^{p'/2} \leqslant A \|g\|_{L^{p'}}^{p'},$$

for some constant $A = A(K)$. So, by the definition of C we have $C \leqslant cA + Ca$ and hence $C < \infty$ uniformly in R as required. □

4.2 Bilinear Kakeya Estimates

This exposition is based on the material found in [4, 27]. Suppose $\mathbb{T}_1, \mathbb{T}_2$ are families of δ-tubes (as defined in conjecture 3.5) in $\mathbb{R}^n$. We allow the tubes within a single family to be parallel. However, we assume that for $j = 1, 2$ the tubes in $\mathbb{T}_j$ have long sides pointing in directions belonging to some sufficiently small *fixed* neighbourhood of two, different, standard basis vectors in S^{n-1}. We say such families of tubes are *transversal.*

Denote by $K(q \times q \to p/2)$ the bilinear Kakeya estimate

$$\left\| \left(\sum_{T \in \mathbb{T}_1} \chi_T \right) \left(\sum_{T \in \mathbb{T}_2} \chi_T \right) \right\|_{L^{p/2}(\mathbb{R}^n)} \lesssim \delta^{\frac{2n}{p} - \frac{2(n-1)}{q'}} (\#\mathbb{T}_1)^{\frac{1}{q}} (\#\mathbb{T}_2)^{\frac{1}{q}}.$$

As for the restriction conjecture there is a bilinear analogue of the Kakeya maximal operator conjecture:

Conjecture 4.12 *(Bilinear Kakeya) If $\frac{1}{p} < \frac{n-1}{n}$ and $\frac{n-2}{q} + \frac{2}{p} \leqslant n-1$ then $K(q \times q \to p/2)$ holds.*

Theorem 4.13 *The above exponents are the best posssible*

Proof. The first bound is obtained in a directly analogous way to that for the linear case in subsection 3.1.2.

To show the necessity of the other bound we adapt the 'stretched caps' example used to show (4.6). If we restrict our attention to the tube $\mathbb{R} \times B_{n-2}$ then we will just be considering, at most, $(\#\mathbb{T})\delta^{n-2} = \delta^{1-n}\delta^{n-2} = \delta^{-1}$ tubes from each family and we will have, also for each family, $\sum\limits_{T\in\mathbb{T}} \chi_T \sim 1$ on a region of volume $\sim \delta^{n-2}$.

Inserting these estimates into $K(q \times q \to p/2)$ yields

$$\delta^{\frac{2(n-2)}{p}} \lesssim \delta^{\frac{2n}{p} - \frac{2(n-1)}{q'}} \delta^{\frac{-2}{q}},$$

$$1 \lesssim \delta^{\frac{2n}{p} - \frac{2(n-1)}{q'} - \frac{2}{q} \frac{2(n-2)}{p}},$$

taking $\delta \to 0$ we obtain

$$\frac{n}{p} - \frac{n-1}{q'} - \frac{1}{q} - \frac{n-2}{p} \leqslant 0,$$

$$\frac{n-2}{q} + \frac{2}{p} \leqslant n-1.$$

□

4.2.1 Wolff's 'Hairbrush' Argument

We progress as in [27]. In this subsection we will use the notation $A \lesssim B$ to denote the estimate $A \leqslant C_\varepsilon \delta^{-\varepsilon} B$.

Theorem 4.14 *For all $n \geqslant 2$ we have*

$$K\left(\frac{n+2}{n+1} \times \frac{n+2}{n+1} \to \frac{n+2}{2n}\right).$$

Proof. We have to show that

$$\left\| \left(\sum_{T\in\mathbb{T}_1} \chi_T\right) \left(\sum_{T\in\mathbb{T}_2} \chi_T\right) \right\|_{L^{\frac{n+2}{2n}}(\mathbb{R}^n)} \lesssim \delta^{\frac{2n}{p} - \frac{2(n-1)}{q'}} (\#\mathbb{T}_1 \#\mathbb{T}_2)^{\frac{n+1}{n+2}},$$

$$\lesssim \delta^{\frac{2n^2}{n+2} - \frac{2(n-1)}{n+2}} (\#\mathbb{T}_1 \#\mathbb{T}_2)^{\frac{n+1}{n+2}},$$

$$\lesssim \delta^{2\frac{n^2-n+1}{n+2}} (\#\mathbb{T}_1 \#\mathbb{T}_2)^{\frac{n+1}{n+2}}. \tag{4.15}$$

It will suffice to show the weak-type bound

$$\left|\left\{\left(\sum_{T\in\mathbb{T}_1}\chi_T\right)\left(\sum_{T\in\mathbb{T}_2}\chi_T\right)\gtrsim\alpha\right\}\right|\lesssim\alpha^{-\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}}(\#\mathbb{T}_1\#\mathbb{T}_2)^{\frac{n+1}{2n}}. \tag{4.16}$$

since (4.15) can be recovered by integrating this over all α of polynomial size. (4.16) will follow from the estimate

$$|E|\lesssim(\alpha_1\alpha_2)^{-\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}}(\#\mathbb{T}_1\#\mathbb{T}_2)^{\frac{n+1}{2n}}, \tag{4.17}$$

where E is defined as

$$E=\{x:\sum_{T_1\in\mathbb{T}_1}\chi_{T_1}(x)>\alpha_1,\sum_{T_2\in\mathbb{T}_2}\chi_{T_2}(x)>\alpha_2\},$$

since

$$\left\{x:\left(\sum_{T\in\mathbb{T}_1}\chi_T(x)\right)\left(\sum_{T\in\mathbb{T}_2}\chi_T(x)\right)>\alpha\right\}\subseteq\bigcup_{1\lesssim 2^{k_1}\lesssim\delta^{-(n-1)}}\left\{x:\sum_{T\in\mathbb{T}_1}\chi_T(x)\gtrsim 2^{k_1},\sum_{T\in\mathbb{T}_2}\chi_T(x)\gtrsim 2^{-k_1}\alpha\right\}$$

(i.e. $\alpha_1=2^{k_1}$ and $\alpha_2=2^{-k_1}\alpha$), and so

$$\begin{aligned}\left|\left\{\left(\sum_{T\in\mathbb{T}_1}\chi_T\right)\left(\sum_{T\in\mathbb{T}_2}\chi_T\right)\gtrsim\alpha\right\}\right| &\lesssim\sum_{1\lesssim\alpha_1\lesssim\delta^{-(n-1)}}\left|\left\{\sum_{T\in\mathbb{T}_1}\chi_T\gtrsim\alpha_1,\sum_{T\in\mathbb{T}_2}\chi_T\gtrsim\frac{\alpha}{\alpha_1}\right\}\right|,\\ &\lesssim\sum_{1\lesssim\alpha_1\lesssim\delta^{-(n-1)}}\alpha^{\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}}(\#\mathbb{T}_1\#\mathbb{T}_2)^{\frac{n+1}{2n}},\\ &\lesssim\alpha^{\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}}(\#\mathbb{T}_1\#\mathbb{T}_2)^{\frac{n+1}{2n}}.\end{aligned}$$

Let

$$\widetilde{\mathbb{T}}_i=\{T_i\in\mathbb{T}_i:\sum_{T_j\in\mathbb{T}_j}|T_j\cap T_i\cap E|>\frac{10^{-d}\alpha_1\alpha_2|E|}{\#\mathbb{T}_j}\},$$

for $i = 1, j = 2$ and $i = 2, j = 1$.

We observe that [1]

$$\int_E \sum_{T_1 \in \widetilde{\mathbb{T}}_1} \chi_{T_1} \sum_{T_2 \in \mathbb{T}_2} \chi_{T_2} \gtrsim \frac{\alpha_1 \alpha_2 |E|}{\# \mathbb{T}_1} \Leftrightarrow \frac{1}{\mathbb{T}_2} \sum_{T_2 \in \mathbb{T}_2} \left(\sum_{T_1 \in \widetilde{\mathbb{T}}_1} |T_1 \cap T_2 \cap E| \right) \gtrsim \frac{\alpha_1 \alpha_2 |E|}{\# \mathbb{T}_1 \# \mathbb{T}_2}.$$

Now we can say

$$\int_E \sum_{T_1 \in \widetilde{\mathbb{T}}_1} \chi_{T_1} \sum_{T_2 \in \widetilde{\mathbb{T}}_2} \chi_{T_2} \gtrsim \alpha_1 \alpha_2 |E|,$$

and this implies

$$\sum_{T_2 \in \widetilde{\mathbb{T}}_2} \left(\int_{T_2 \cap E} \sum_{T_1 \in \widetilde{\mathbb{T}}_1} \chi_{T_1} \right) \gtrsim \alpha_1 \alpha_2 |E|.$$

Hence there exists a tube $T_2^0 \in \widetilde{\mathbb{T}}_2$ (the 'handle' of the brush) such that

$$\int_{T_2^0 \cap E} \sum_{T_1 \in \widetilde{\mathbb{T}}_1} \chi_{T_1} \gtrsim \frac{\alpha_1 \alpha_2 |E|}{\# \widetilde{\mathbb{T}}_2},$$

$$\int_{T_2^0} \sum_{T_1 \in \widetilde{\mathbb{T}}_1} \chi_{T_1} \gtrsim \frac{\alpha_1 \alpha_2 |E|}{\# \mathbb{T}_2}.$$

By affine invariance we can take the handle as the vertical tube through the origin.

Now let

$$\widetilde{\mathbb{T}}_1^0 = \{T_1 \in \widetilde{\mathbb{T}}_1 : T_1 \cap T_2^0 \neq \emptyset\},$$

(these are the 'bristles' of the brush). Then we can observe that

$$|T_1 \cap E| \gtrsim \lambda_1 \delta^{n-1}, \tag{4.18}$$

for all $T_1 \in \widetilde{\mathbb{T}}_1^0$; and that

$$\#\widetilde{\mathbb{T}}_1^0 \gtrsim \delta^{-1}\alpha_1\lambda_2, \tag{4.19}$$

where

$$\lambda_j = \frac{\alpha_j|E|}{\#\mathbb{T}_j\delta^{n-1}}$$

for $j = 1, 2$. Let β range dyadically between $\lambda_1 \lesssim \beta \lesssim 1$ and for each such beta let Γ_β be the cylindrical region

$$\Gamma_\beta = \{(\underline{y}, y_n) : |\underline{y}| \sim \beta\}.$$

Thus we can rewrite (4.18):

$$\sum_{\lambda_1 \lesssim \beta \lesssim 1} |T_1 \cap E \cap \Gamma_\beta| \gtrsim \lambda_1\delta^{n-1}$$

for all $\mathbb{T}_1 \in \widetilde{\mathbb{T}}_1^0$.

Now, we have $1 \leqslant \alpha_1, \alpha_2 \leqslant \delta^{-(n-1)}$ since $1 \leqslant \#\mathbb{T}_1, \#\mathbb{T}_2 \leqslant \delta^{-(n-1)}$ so

$$(\alpha_1\alpha_2)^{-\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}} \geqslant \delta^{(n-1)\left(\frac{n+2}{2n}\right)-\frac{n^2-n+1}{n}}.$$

If $|E| \leqslant \delta^{n^{10}}$, then

$$|E| \leqslant \delta^{(n-1)\left(\frac{n+2}{2n}\right)-\frac{n^2-n+1}{n}} \leqslant (\alpha_1\alpha_2)^{-\frac{n+2}{2n}}\delta^{\frac{n^2-n+1}{n}}$$

and there would be nothing to prove, so without loss of generality we can assume $|E| > \delta^{n^{10}}$.

We can therefore say that, since $\#\mathbb{T}_1\delta^{n-1} \leqslant 1$,

$$\lambda_1 = \frac{\alpha_1|E|}{\#\mathbb{T}_1\delta^{n-1}} \geqslant \delta^{n^{10}},$$

and so there are $O(\log(1/\lambda_1))$ such dyadic βs, and therefore there exists a β such that

$$|T_1 \cap E \cap \Gamma_\beta| \gtrsim \frac{\lambda_1 \delta^{n-1}}{\log(1/\lambda_1)},$$

which is the same as

$$|T_1 \cap E \cap \Gamma_\beta| \gtrsim \lambda_1 \delta^{n-1}. \tag{4.20}$$

Further, we can refine $\widetilde{\mathbb{T}}_1^0$ so that (4.20) holds for all T_1 in the refined $\widetilde{\mathbb{T}}_1^0$; henceforth this β is considered fixed.

The tubes in $\widetilde{\mathbb{T}}_1^0$ are δ-separated in direction. It will suit our purposes to work with a set of directions of greater separation: we define $\widehat{\mathbb{T}}_1^0$ to be any δ/β-separated subset of $\widetilde{\mathbb{T}}_1^0$. Note that we have the estimates

$$\#\widehat{\mathbb{T}}_1^0 \gtrsim \beta^{n-1} \#\widetilde{\mathbb{T}}_1^0 \tag{4.21}$$

and

$$\beta \gtrsim \lambda_1. \tag{4.22}$$

Let Θ be a δ/β-net of the unit sphere S^{n-2} in $\mathbb{R}^{n-1}$. For each $T \in \widehat{\mathbb{T}}_1^0$, we can isolate an element $\theta = \theta_T$ of Θ by imposing the condition

$$|\theta - \frac{\omega}{|\omega|}| \lesssim \delta/\beta \tag{4.23}$$

where ω is the angular separation of T from the basis vector to which the directions of the family $\mathbb{T}_1$ belong to a neighbourhood of. Recall that $|\omega| \sim 1$ for all $T \in \widehat{\mathbb{T}}_1^0$. From simple geometrical considerations and since each $T \in \widetilde{\mathbb{T}}_1^0$ intersects T_2^0 we see that, for each $T \in \widetilde{\mathbb{T}}_1^0$, $T \cap \mathit{\Gamma}_\beta$ is contained in the slab Π_θ given by

$$\Pi_\theta = \{(\underline{y}, y_n : |\underline{y}| \sim \beta, |\frac{\underline{y}}{|\underline{y}|} - \theta| \lesssim \delta/\beta\}.$$

Define $\widehat{\mathbb{T}}_{1,\theta}^0$ as

$$\widehat{\mathbb{T}}_{1,\theta}^0 = \{T \in \widehat{\mathbb{T}}_1^0 : \theta_T = \theta\}.$$

We consider the quantity

$$Q = \int\limits_{E\cap\Pi_\theta} \sum_{T\in\widehat{\mathbb{T}}_{1,\theta}^0} \chi_T.$$

We will estimate Q in two different ways: firstly, from the fact that, for each $T \in \widetilde{\mathbb{T}}_1^0$, $T \cap \mathit{\Gamma}_\beta$ is contained in the slab Π_θ and (4.20) we have

$$|T_1 \cap E \cap \Pi_\theta| \gtrsim \lambda_1 \delta^{n-1}$$

for all $T_1 \in \widehat{\mathbb{T}}_{1,\theta}^0$. If we sum this over the tubes $\widehat{\mathbb{T}}_{1,\theta}^0$ we have

$$Q \gtrsim \#\widehat{\mathbb{T}}_{1,\theta}^0 \lambda_1 \delta^{n-1}. \tag{4.24}$$

We can also estimate Q using the Cauchy-Schwarz inequality:

$$Q \lesssim |E \cap \Pi_\theta|^{1/2} \left(\int\limits_{|E\cap\Pi_\theta|} \left(\sum_{T\in\widehat{\mathbb{T}}_{1,\theta}^0} \chi_T \right)^2 \right)^{1/2}.$$

If we now square both sides and rewrite the integrand as the sum of the diagonal and off-diagonal term we have

$$\frac{Q^2}{|E\cap\Pi_\theta|}\lesssim\left(\int_{|E\cap\Pi_\theta|}\sum_{T\in\widehat{\mathbb{T}}^0_{1,\theta}}\chi_T\right)+\sum_{T\in\widehat{\mathbb{T}}^0_{1,\theta}}\sum_{T\neq T'}|T\cap T'\cap E\cap\Pi_\theta|. \tag{4.25}$$

We observe that the first term on the right is just Q. By some elementery geometry [20] we have

$$|T\cap T'|\lesssim\frac{\delta^n}{|\omega-\omega'|+\delta},$$

where ω and ω' are the directions of T and T'. Thus (4.25) becomes

$$\frac{Q^2}{|E\cap\Pi_\theta|}\lesssim Q+\sum_{T\in\widehat{\mathbb{T}}^0_{1,\theta}}\sum_{T\neq T'}\frac{\delta^n}{|\omega-\omega'|}.$$

Now, we can place an upper bound on the number of tubes from one indexing set, T', that have angular separation less than a given parameter from a member of the other indexing set T. This comes from our condition (4.23): for each T, the number of T' such that $|\omega-\omega'|\sim 2^{-j}$ is at most $1/(2^j\delta/\beta)$ for any j. If we let 2^j range over $\delta/\beta\lesssim 2^j\lesssim 1$ we can rewrite the above estimate, involving a double summation, as one involving just one:

$$\frac{Q^2}{|E\cap\Pi_\theta|}\lesssim Q+\sum_{\delta/\beta\lesssim 2^j\lesssim 1}\#\widehat{\mathbb{T}}^0_{1,\theta}\frac{1}{2^j\delta/\beta}\frac{\delta^n}{2^{-j}},$$

and since the number of such j is logarithmic, this is simply

$$\frac{1}{|E\cap\Pi_\theta|}\lesssim\frac{1}{Q}+\frac{\#\widehat{\mathbb{T}}^0_{1,\theta}\beta\delta^{n-1}}{Q^2}.$$

Combining this with (4.24) we have

$$\frac{1}{|E\cap\Pi_\theta|}\lesssim\frac{\#\mathbb{T}_1}{\alpha_1|E|\#\widehat{\mathbb{T}}^0_{1,\theta}}+\frac{\#\mathbb{T}_1^2\widehat{\mathbb{T}}^0_{1,\theta}\beta\delta^{n-1}}{\alpha_1^2|E|^2(\widehat{\mathbb{T}}^0_{1,\theta})^2},$$

$$\frac{1}{|E \cap \Pi_\theta|} \lesssim \frac{\alpha_1 |E| \# \mathbb{T}_1 + \# \mathbb{T}_1^2 \beta \delta^{n-1}}{\alpha_1^2 |E|^2 \# \widehat{\mathbb{T}}_{1,\theta}^0}.$$

Using the hypothesis (4.22) we obtain

$$|E \cap \Pi_\theta| \gtrsim \frac{\alpha_1^2 |E|^2 \# \widehat{\mathbb{T}}_{1,\theta}^0}{\# T_1^2 \beta \delta^{n-1}},$$

$$|E \cap \Pi_\theta| \gtrsim \frac{\lambda_1^2 \# \widehat{\mathbb{T}}_{1,\theta}^0 \delta^{n-1}}{\beta}.$$

Since the Π_θ are essentially disjoint we have

$$|E| \gtrsim \frac{\lambda_1^2 \# \widehat{\mathbb{T}}_1^0 \delta^{n-1}}{\beta}.$$

From (4.21) and (4.22) we have

$$|E| \gtrsim \beta^{n-2} \lambda_1^2 \# \widetilde{\mathbb{T}}_1^0 \delta^{n-1},$$

$$|E| \gtrsim \# \widetilde{\mathbb{T}}_1^0 \lambda_1^n \delta^{n-1},$$

or

$$|E| \gtrsim \# \widetilde{\mathbb{T}}_1^0 \left(\frac{\alpha_1 |E|}{\# \mathbb{T}_1} \right)^n \delta^{-(n-1)^2}.$$

Combining with (4.19) this yields

$$|E| \gtrsim \frac{\alpha_1^{n+1} \alpha_2 |E|^{n+1} \delta^{-(n-1)^2 - n}}{(\# \mathbb{T}_1)^n \# \mathbb{T}_2}.$$

By a completely symmetrical argument we arrive at

$$|E| \gtrsim \frac{\alpha_2^{n+1} \alpha_1 |E|^{n+1} \delta^{-(n-1)^2 - n}}{(\# \mathbb{T}_2)^n \# \mathbb{T}_1}.$$

If we take the geometric mean of the last two estimates and rearrange we, finally, have

$$|E| \lesssim (\#\mathbb{T}_1 \#\mathbb{T}_2)^{\frac{n+1}{2n}} (\alpha_1 \alpha_2)^{\frac{n+2}{2n}} \delta^{\frac{n^2-n+1}{n}}. \qquad \square$$

Chapter 5

The Latest Developments in the Field

As stated in the introduction, the role of curvature in the restriction problem has been central since Stein's observations in the 1960s. The bilinear approach of the 1990s sought to exploit this role in a more geometric way than had been attempted previously by incorporating the concept of *transversality*. Indeed, for $n = 2$, only transversality is required for the bilinear restriction problem: curvature is no longer required (see [3]). However, for $n > 3$, the roles of curvature and transversality are 'intertwined' and we can no longer dispense with the curvature hypothesis. This naturally leads, see [4], to a *multi*-linear approach, i.e. k-linear, where $2 \leqslant k \leqslant n$.

For the case $k = n$, i.e. the n-linear case we have the following analogue of the bilinear restriction conjecture:

Conjecture 5.1 *(Multilinear restriction) If $S_1, \ldots, S_n$ are transversal then for $\frac{1}{q} \leqslant \frac{n-1}{2n}$ and $\frac{1}{q} \leqslant \frac{n-1}{n}\frac{1}{p'}$ we have*

$$\|\widehat{f_1 d\omega} \cdots \widehat{f_n d\omega}\|_{L^{\frac{q}{n}}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^p(S_1)} \cdots \|f_n\|_{L^p(S_n)}.$$

Note that curvature does not feature in this conjecture.

The following 'weak' version of this conjecture was proved in [4]

Theorem 5.2 *(Near optimal multilinear restriction) With the same hypotheses as Conjecture 5.1, for each $\varepsilon > 0$ we have*

$$\|\widehat{f_1 d\omega} \cdots \widehat{f_n d\omega}\|_{L^{\frac{q}{n}}(B(0,R))} \lesssim R^{\varepsilon} \|f_1\|_{L^p(S_1)} \cdots \|f_n\|_{L^p(S_n)},$$

for all $R \geqslant 1$.

A slight refinement of this theorem is, see [3],

Theorem 5.3 *(Refinement of theorem 5.2) With the same hypotheses as theorem 5.2 there exists a constant $\kappa < \infty$ such that*

$$\|\widehat{f_1 d\omega} \cdots \widehat{f_n d\omega}\|_{L^{\frac{q}{n}}(B(0,R))} \lesssim (\log R)^{\kappa} \|f_1\|_{L^p(S_1)} \cdots \|f_n\|_{L^p(S_n)},$$

for all $R \geqslant 1$.

Similarly there is a multilinear analogue of the Kakeya maximal operator conjecture:

Theorem 5.4 *(Multilinear Kakeya) If $\frac{n}{n-1} \leqslant p \leqslant \infty$ then, for transversal families of tubes $\mathbb{T}_1, \ldots, \mathbb{T}_n$ we have*

$$\left\| \prod_{j=1}^{n} \left(\sum_{T \in \mathbb{T}_j} \chi_T \right) \right\|_{L^{\frac{p}{n}}(\mathbb{R}^n)} \lesssim \prod_{j=1}^{d} (\delta^{\frac{n}{p}} \# \mathbb{T}_j).$$

This was proved up to the end-point in [4] and for the end-point in [19] and [14]. As was shown in subsection 3.3 the linear restriction conjecture implies the linear Kakeya maximal operator conjecture. It is not known if the reverse implication holds but, interestingly, it was shown in [4] that at the n-linear level theorems 5.3 and 5.4 are equivalent. The same Rademacher-function argument can be applied to show that the n-linear restriction conjecture implies the n-linear Kakeya maximal operator conjecture.

Until 2010 the best linear restriction estimates were obtained via bilinear ones. In [12] Bourgain and Guth devised a mechanism for which linear estimates can be obtained from multilinear ones and which is able to use input from progress on the Kakeya problem (i.e. Wolff's best Kakeya estimate). For instance, following a similar argument to that in the proof of theorem 4.9 they were able, using the trilinear restriction conjecture in three dimensions, to show that, for $\mathcal{C}$ the best constant in

$$\|\widehat{f\mathrm{d}\omega}\|_{L^q(B(0,R))} \leqslant C\|f\|_{L^p(\mathrm{d}\omega)},$$

we have

$$\mathcal{C} \leqslant c_3 K^{power} + c_2(K')^{power} K^{\frac{1}{2}-\frac{1}{q}} K^{\frac{6}{q}-2}\mathcal{C} + c_1(K')^{\frac{6}{q}-2}\mathcal{C},$$

where c_1, c_2 and c_3 are constants and K, K' are sizes of caps. Taking K' and K sufficiently large gives $\mathcal{C} < \infty$ uniformly in $R \gg 1$ for $p \geqslant q$ and $q > 3.3$. This, to date, is the best linear estimate in 3 dimensions.

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